Euclidean Space

Let $\mathbb{R}^n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{R} \ \forall i\}$.

We denote its elements by

$\vec{x} := (x_1, x_2, \dots, x_n)$.

When $n=2$ or $n=3$, we often just write (x,y) or (x,y,z).

We write $\vec{0} := (0,0,\dots,0)$ for the zero vector in $\mathbb{R}^n$.

Inner product structure on $\mathbb{R}^n$

Addition: $\vec{x}+\vec{y} = (x_1+y_1, x_2+y_2, \dots, x_n+y_n) \ \forall \vec{x},\vec{y} \in \mathbb{R}^n$.

Scalar multiplication: $c\vec{x} = (cx_1, cx_2, \dots, cx_n)$, $c \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^n$.

Inner product: $\langle \vec{x},\vec{y}\rangle = x_1y_1 + \dots + x_ny_n \ \forall \vec{x},\vec{y} \in \mathbb{R}^n$.

Norm: $\|\vec{x}\| = \sqrt{\langle \vec{x},\vec{x}\rangle}$, $\vec{x} \in \mathbb{R}^n$.

The norm function $\|\cdot\| \to [0,\infty)$ satisfies the 3 properties of a norm:

i) $\|\vec{x}\| = 0$ iff. $\vec{x} = \vec{0}$.

Proof:

$$\|\vec{x}\| = 0 = \sqrt{x_1^2+\dots+x_n^2} \Leftrightarrow 0 = x_1^2+\dots+x_n^2$$
$$\Leftrightarrow x_i^2 = 0 \ \forall i = 1,\dots,n$$
$$\Leftrightarrow \vec{x} = \vec{0}. \qquad \square$$

ii) $\|c\vec{x}\| = |c|\,\|\vec{x}\| \ \forall c \in \mathbb{R}, \vec{x} \in \mathbb{R}^n$

Proof: exercise.

iii) $\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\| \ \forall \vec{x},\vec{y} \in \mathbb{R}^n$ (Δ inequality)

To prove (iii) we use the Cauchy-Schwarz inequality.

Cauchy-Schwarz inequality

$|\langle \vec{x},\vec{y}\rangle| \le \|\vec{x}\| \cdot \|\vec{y}\| \ \forall \vec{x},\vec{y} \in \mathbb{R}^n$

Recall: $|\langle \vec{x}, \frac{\vec{y}}{\|\vec{y}\|}\rangle|$ for $\vec{y} \ne \vec{0}$ is the length of the orthogonal projection of $\vec{x}$ onto $\mathbb{R}\vec{y}$ (the line spanned by $\vec{y}$).

Proof:

If $\vec{y} = \vec{0}$ then

$$|\langle \vec{x},\vec{y}\rangle| = |\langle \vec{x},\vec{0}\rangle| = 0 = \|\vec{x}\|\,\|\vec{y}\|$$

Suppose $\vec{y} \ne \vec{0}$, and consider the quadratic function

$$f(t) = \|\vec{x} - t\vec{y}\|^2 = \langle (\vec{x}-t\vec{y}), (\vec{x}-t\vec{y})\rangle$$
$$= \langle \vec{x},\vec{x}\rangle - t\langle \vec{x},\vec{y}\rangle - t\langle \vec{y},\vec{x}\rangle + t^2\langle \vec{y},\vec{y}\rangle$$
$$= \|\vec{x}\|^2 - 2t\langle \vec{x},\vec{y}\rangle + t^2\|\vec{y}\| \qquad [\ge 0 \ \forall t \in \mathbb{R}]$$

Minimum of f occurs when $f'(t) = 0$, or

$$0 = f'(t) = -2\langle \vec{x},\vec{y}\rangle + 2t\|\vec{y}\|^2 \Rightarrow t = \frac{\langle \vec{x},\vec{y}\rangle}{\|\vec{y}\|^2}$$
$$\Rightarrow 0 \le f\left(\frac{\langle \vec{x},\vec{y}\rangle}{\|\vec{y}\|^2}\right) = \|\vec{x}\|^2 - \frac{2\langle \vec{x},\vec{y}\rangle^2}{\|\vec{y}\|^2} + \frac{\langle \vec{x},\vec{y}\rangle^2}{\|\vec{y}\|^2} = \|\vec{x}\|^2 - \frac{\langle \vec{x},\vec{y}\rangle^2}{\|\vec{y}\|^2}$$
$$\Rightarrow \frac{\langle \vec{x},\vec{y}\rangle^2}{\|\vec{y}\|} \le \|\vec{x}\|^2$$
$$\Rightarrow \langle \vec{x},\vec{y}\rangle^2 \le \|\vec{x}\|^2\|\vec{y}\|^2$$
$$\Rightarrow \langle \vec{x},\vec{y}\rangle \le \|\vec{x}\|\,\|\vec{y}\|. \qquad \square$$