MATH 2001 — Lecture 2 — 11 September 2026

Recall:

For $\vec{x}, \vec{y} \in \mathbb{R}^n$, the angle $\theta \in [0, \pi]$ between them satisfies

$$\theta = \arccos\left(\frac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{x}\| \|\vec{y}\|}\right)$$

$\theta = \frac{\pi}{2} \Leftrightarrow \langle \vec{x}, \vec{y} \rangle = 0$. In this case, we say that $\vec{x}$ and $\vec{y}$ are orthogonal.

Corollary ($\triangle$ inequality)

For any $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|.$$

Proof:

$$\begin{aligned} \|\vec{x}+\vec{y}\|^2 &= \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle \\ &= \|\vec{x}\|^2 + 2\langle \vec{x}, \vec{y} \rangle + \|\vec{y}\|^2. \end{aligned}$$

But $\langle \vec{x}, \vec{y} \rangle \leq |\langle \vec{x}, \vec{y} \rangle|$ (since $a \leq |a| \ \forall a \in \mathbb{R}$)

$\leq \|\vec{x}\| \cdot \|\vec{y}\|$ (using the Cauchy-Schwarz inequality)

$$\begin{aligned} \Rightarrow \|\vec{x} + \vec{y}\|^2 &\leq \|\vec{x}\|^2 + 2\langle \vec{x}, \vec{y} \rangle + \|\vec{y}\|^2 \\ &= (\|\vec{x}\| + \|\vec{y}\|)^2. \end{aligned}$$

Taking square roots, $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$. $\square$

Remark:

While the $\triangle$ inequality is a handy tool in proofs, it is typically a "bad" estimate of the size of $\|\vec{x} + \vec{y}\|$.

e.g. if $\vec{x} = (10^6, 0)$, $\vec{y} = (-10^6, \varepsilon)$, $0 < \varepsilon \ll 1$.

$\Rightarrow \vec{x} + \vec{y} = (0, \varepsilon) \Rightarrow \|\vec{x} + \vec{y}\| = \varepsilon$.

But $\|\vec{x}\| + \|\vec{y}\| = 10^6 + \sqrt{(10^6)^2 + \varepsilon^2} \gg \varepsilon$.

Geometry of $\mathbb{R}$-valued functions

Recall that the graph of $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is

$$G(f) = \{(x, f(x)) \mid x \in I\} \subseteq \mathbb{R}^2.$$

If $U \subseteq \mathbb{R}^n$, and $f: U \to \mathbb{R}$, its graph is

$$G(f) = \{(x_1, \ldots, x_n, f(x_1, \ldots, x_n)) \in \mathbb{R}^{n+1} \mid (x_1, \ldots, x_n) \in U\} \subseteq \mathbb{R}^{n+1}.$$

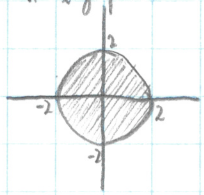
In particular, when $n = 2$, $G(f) = \{(x, y, f(x,y)) \mid (x,y) \in U\}$ is a surface in $\mathbb{R}^3$.

Ex.

$U = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$, $f(x,y) = x^2 + y^2$.

$\|(x,y)\|^2 \leq 4 \Leftrightarrow \|(x,y)\| \leq 2$.

In xy-plane:



In $\mathbb{R}^3$: I can't draw the 3D Desmos graphs, but it can be reproduced on Desmos.

Ex.

$f(x,y) = x^2 - y^2$. We study $z = f(x,y) = x^2 - y^2$ by fixing one of x, y, z and studying the the relationship between the remaining variables.

e.g. $y = 0 \Rightarrow z = x^2$ (regular parabola)

$y = 2 \Rightarrow z = x^2 - 4$

$x = 0 \Rightarrow z = -y^2$

$z = 1 \Rightarrow 1 = x^2 - y^2$ (level curve), $\{(x,y) \mid 1 = x^2 - y^2\}$ hyperbola.

Subsets of Euclidean

Def.

The open disk of radius $r > 0$ centred at $\vec{x} \in \mathbb{R}^n$ is the set

$D_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^n \mid \|\vec{x} - \vec{y}\| < r\}$.

Ex.

$n = 2$, $\vec{x} = (a, b)$, $\vec{y} = (x, y) \in D_r(\vec{x})$ iff. $\|\vec{x} - \vec{y}\|^2 = (a-x)^2 + (b-y)^2 < r^2$

$\Leftrightarrow (x-a)^2 + (y-b)^2 < r^2$

When $n = 3$, $\vec{x} = (a, b, c)$, $\vec{y} = (x, y, z) \in D_r(\vec{x})$

iff. $\underbrace{(x-a)^2 + (y-b)^2 + (z-c)^2}_{\|\vec{x} - \vec{y}\|} < r^2$

The values of the open disk form a sphere around the point (a, b, c).

For $n = 1$, $y \in D_r(x) \Leftrightarrow |x - y| < r \Leftrightarrow -r < y - x < r \Leftrightarrow x - r < y < x + r$.

So $D_r(x) = (x - r, x + r)$.