MATH 2001 Lecture 3 16 September 2026

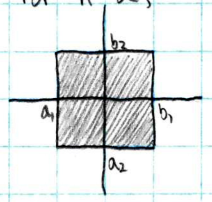
Recall:

We defined the open disk of radius $r > 0$
$D_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^n \mid \|\vec{x} - \vec{y}\| < r\}$
centred at $\vec{x} \in \mathbb{R}^n$.
For $n = 1$, $D_r(x) = (x - r, x + r)$.
For $n = 2$, $D_r(a, b) = \{(x, y) \mid (x-a)^2 + (y-b)^2 < r\}$. (open disk).
For $n = 3$, $D_r(\vec{x})$ = open ball.

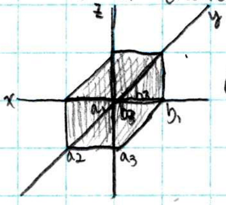
Another class of subsets of interest are rectangular boxes.
$R = [a_1, b_1] \times [a_2, b_2] \times \dots \times [a_n, b_n]$, $a_i, b_i \in \mathbb{R}$, $a_i < b_i$ $\forall\, 1 \le i \le n$.
Recall that if $A, B \subseteq \mathbb{R}$, $A \times B = \{(x, y) \in \mathbb{R}^2 \mid x \in A,\ y \in B\}$

Ex.

For $n = 2$,



For $n = 3$, it creates a rectangular prism.

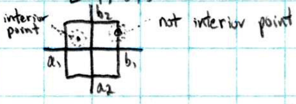


$R = [\underset{x}{a_1, b_1}] \times [\underset{y}{a_2, b_2}] \times [\underset{z}{a_3 \times b_3}]$
Crude diagram.

Def.

Let $A \subseteq \mathbb{R}^n$.

- A is <u>bounded</u> if $\exists M > 0$ s.t. $\|\vec{x}\| \le M$ $\forall \vec{x} \in A$.
  eg. disks, rectangular boxes are bounded, but $\mathbb{R} \times \{0\} = \{(x, 0) \mid x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ is not bounded.
- The <u>complement</u> of A, denoted A^c, is $A^c = \{\vec{x} \in \mathbb{R}^n \mid x \notin A\}$.
  eg. $A = D_r(\vec{x})$. Then $A^c = \{\vec{y} \in \mathbb{R}^n \mid \|\vec{x} - \vec{y}\| \ge r\}$.
- A point $\vec{x} \in A$ is an <u>interior point</u> of A if $\exists r > 0$ s.t. $D_r(\vec{x}) \subseteq A$.
  We denote the subset of interior points by A°.
  eg. $A = [a_1, b_1] \times [a_2, b_2]$



- A is <u>open</u> if $A = A^\circ$. That is, $\forall \vec{x} \in A$, $\exists r > 0$ $(r = r_{\vec{x}})$ s.t. $D_r(\vec{x}) \subseteq A$ (e.g. $A = \emptyset$ is open, $A = \mathbb{R}^n$ is open)
- A is <u>closed</u> if A^c is open.
  eg. $A = \mathbb{R}^n$ is closed as $A^c = \emptyset$ is open. $A = \emptyset$ is closed as $A^c = \mathbb{R}^n$ is open.

Fact:

$\mathbb{R}^n$ and $\emptyset$ are the <u>only</u> subsets of $\mathbb{R}^n$ that are simultaneously open and closed.

Ex.

An open disk $D_r(\vec{x})$ is an open subset of $\mathbb{R}^n$.

Proof:

$\vec{x} \in \mathbb{R}^n$, $r > 0$ are fixed.

Need to show $\forall \vec{y} \in D_r(\vec{x})$, $\exists\, r_{\vec{y}} > 0$ st. $D_{r_{\vec{y}}}(\vec{y}) \subseteq D_r(\vec{x})$.

Pick $r_{\vec{y}} = r - \|\vec{x} - \vec{y}\|$. Then $\vec{r}_y > 0$ (as $\vec{y} \in D_r(\vec{x})$.)

Need to show that every $z \in D_{r_{\vec{y}}}(\vec{y})$ is contained in $D_r(\vec{x})$.

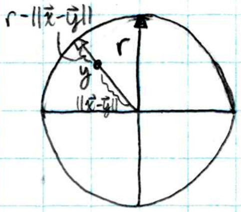
Indeed, if $z \in D_{r_{\vec{y}}}(\vec{y})$, that is,

$$\|\vec{y} - \vec{z}\| < r_{\vec{y}} = r - \|\vec{x} - \vec{y}\|$$

$$\Rightarrow \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| < r.$$

By $\triangle$ inequality, $\|\vec{x} - \vec{z}\| = \|(\vec{x} - \vec{y}) + (\vec{y} - \vec{z})\| \leq \|\vec{x} - \vec{y}\| + \|\vec{y} - \vec{z}\| < r.$

Thus, $z \in D_r(\vec{x})$. ☐



Ex.

$A = \{(x,y) \in \mathbb{R}^2 \mid x > 0\}$ is open.

Need to show that $\forall (x,y) \in A$, $\exists\, r > 0$ st. $D_r((x,y)) \subseteq A$.

Pick $r = x > 0$. Then for any $(a,b) \in D_r((x,y))$,

$$\|(a,b) - (x,y)\| = \sqrt{(a-x)^2 + (b-y)^2} < r = x.$$

Need to conclude $a > 0$ so that $(a,b) \in A$.

Squaring both sides,

$$(a-x)^2 + (b-y)^2 < x^2$$

$$\Rightarrow (a-x)^2 \leq (a-x)^2 + (b-y)^2 < x^2 \qquad (\text{since } (b-y)^2 \geq 0)$$

$$\Rightarrow (a-x)^2 < x^2$$

$$\Rightarrow |a-x| < x$$

$$\Rightarrow -x < a - x < x$$

$$\Rightarrow 0 < a < 2x$$

$$\Rightarrow a > 0.$$ ☐