Def.

Let $A \subseteq \mathbb{R}^n$. A point $\vec{x} \in \mathbb{R}^n$ is a <u>boundary point</u> of A if $\forall r > 0$, $D_r(\vec{x}) \cap A,\ D_r(\vec{x}) \cap A^c \neq \emptyset$.

(Every disk about $\vec{x}$ meets both A and A^c.)

The set of boundary points of A is denoted ∂A.

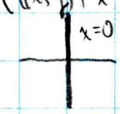
Exercise

A is closed $\Leftrightarrow \partial A \subseteq A$.

Ex.

(1) $\partial D_r(\vec{x}) = \{\vec{y} \in \mathbb{R}^n \mid \|\vec{x} - \vec{y}\| = r\}$

(2) $\partial(\{(x,y) \mid x > 0\}) = \{(0,y) \mid y \in \mathbb{R}\}$



Proof of $\partial D_r(\vec{x}) = S_r(\vec{x}) := \{\vec{y} \in \mathbb{R}^n \mid \|\vec{x} - \vec{y}\| = r\}$.

<u>$\partial D_r(\vec{x}) \subseteq S_r(\vec{x})$:</u>

take $\vec{y} \in \partial D_r(\vec{x})$. Then $\forall \varepsilon > 0$,

$D_\varepsilon(\vec{y}) \cap D_r(\vec{x}) \neq \emptyset$ and $D_\varepsilon(\vec{y}) \cap D_r(\vec{x})^c \neq \emptyset$.

Rough work: take $\vec{z} \in D_\varepsilon(\vec{y}) \cap D_r(\vec{x})$. $\|\vec{z} - \vec{y}\| < \varepsilon$, $\|\vec{z} - \vec{x}\| < r$

$\Rightarrow \|\vec{x} - \vec{y}\| \leq \|\vec{x} - \vec{z}\| + \|\vec{z} - \vec{y}\| < r + \varepsilon$

$\Rightarrow \forall \varepsilon > 0$, $\|\vec{x} - \vec{y}\| < r + \varepsilon$

$\Rightarrow \|\vec{x} - \vec{y}\| \leq r$ $\quad$ ($a, b \in \mathbb{R}$ $a < b + \varepsilon$ $\forall \varepsilon > 0 \Leftrightarrow a \leq b$).

[Method: $\forall a, b \in \mathbb{R}$, $a = b \Leftrightarrow a \leq b$ and $b \leq a \Leftrightarrow \forall \varepsilon > 0$, $a < b + \varepsilon$, $b < a + \varepsilon$]

Next, $\exists\ \vec{z}_c \in D_\varepsilon(\vec{y}) \cap D_r(\vec{x})^c$, meaning $\|\vec{z}_c - \vec{y}\| < \varepsilon$, $\|\vec{z}_c - \vec{x}\| \geq r$.

$\Rightarrow r \leq \|\vec{z}_c - \vec{x}\| \leq \|\vec{z}_c - \vec{y}\| + \|\vec{y} - \vec{x}\| < \varepsilon + \|\vec{y} - \vec{x}\|$

$\Rightarrow \forall \varepsilon > 0$, $\|\vec{y} - \vec{x}\| > r - \varepsilon$

$\Rightarrow \|\vec{y} - \vec{x}\| \geq r$

Thus $\|\vec{x} - \vec{y}\| = r$ and $\vec{y} \in S_r(\vec{x})$

Proof of $S_r(x) \subseteq \partial D_r(\vec{x})$ is similar but omitted.

<u>Limits and Continuity</u>

Unless otherwise stated, $A \subseteq \mathbb{R}^n$ is open.

Def.

Let $f: A \to \mathbb{R}$, $\vec{a} \in A \cup \partial A$. (eg. if $A = \mathbb{R}^2 \setminus \{(0,0)\}$, $\vec{a} \in \mathbb{R}^2$ since $\partial A = \mathbb{R}^2$)

- $b \in \mathbb{R}$ is the limit of $f(\vec{x})$ as $\vec{x}$ approaches $\vec{a}$, written $\lim_{\vec{x} \to \vec{a}} f(\vec{x}) = b$, if $\varepsilon > 0$, $\exists \delta > 0$ ($\delta = \delta_{\vec{a}, \varepsilon, f}$) s.t. $|f(\vec{x}) - b| < \varepsilon$ whenever $0 < \|\vec{x} - \vec{a}\| < \delta$.

- If $f(\vec{a})$ and $\lim_{\vec{x}\to\vec{a}} f(\vec{x})$ both exist and
  $f(\vec{a}) = \lim_{\vec{x}\to\vec{a}} f(\vec{x})$ $\quad \left(f\left(\lim_{\vec{x}\to\vec{a}} \vec{x}\right) = \lim_{\vec{x}\to\vec{a}} f(\vec{x})\right)$
  we say that f is <u>continuous</u> (cts) at $\vec{a}$.

Thus, f is cts. at $\vec{a}$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|f(\vec{x}) - f(\vec{a})| < \varepsilon$ whenever $\|\vec{x} - \vec{a}\| < \delta$.

<u>Ex.</u>

The norm function $f(\vec{x}) = \|\vec{x}\|$ is cts. $(f: \mathbb{R}^n \to \mathbb{R})$.
Given $\vec{a} \in \mathbb{R}^n$, $\varepsilon > 0$, need to find $\delta > 0$ s.t. $|\|\vec{x}\| - \|\vec{a}\|| < \varepsilon$ whenever $\|\vec{x} - \vec{a}\| < \delta$.
(side note: by reverse Δ inequality, $|\|\vec{x}\| - \|\vec{a}\|| \le \|\vec{x} - \vec{a}\|$)
Choosing $\delta = \varepsilon$, whenever $\|x - a\| < \delta$,
$|\|\vec{x}\| - \|\vec{a}\|| \le \|\vec{x} - \vec{a}\| < \delta = \varepsilon$.

☆ no matter how you approach $\vec{a}$ in domain, (via lines or other curves) $f(\vec{x})$ must approach some value $f(\vec{a})$ (to be cts. at $\vec{a}$).

<u>Ex.</u>

Let $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \ne (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$ Show $\lim_{(x,y)\to(0,0)} f(x,y)$ DNE.
Approaching $(0,0)$ along the line $y = mx$, $m \in \mathbb{R}$, for $x \ne 0$,
$$f(x, mx) = \frac{x(mx)}{x^2 + (mx)^2} = \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$$
$\Rightarrow$ Approaching $(0,0)$ along different straight lines $y = mx$ (varying $m \in \mathbb{R}$) yields different limits:
$\lim_{x\to 0} f(x, mx) = \frac{m}{1+m^2}$.