Ex. Find the partial fraction decomposition of $\frac{6x-3}{x^3-1}$

$$\frac{6x-3}{x^3-1} = \frac{6x-3}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

(note: x^2+x+1 is not reducible in $\mathbb{R}$)

$\Rightarrow 6x-3 = A(x^2+x+1) + (Bx+C)(x-1)$ (multiply by denominator)

let $x=1 \Rightarrow 3 = 3A \Rightarrow A = 1.$

$x=0 \Rightarrow -3 = 1 + C(-1) \Rightarrow C = 4$ (choose x strategically to try isolating values)

$x=2 \Rightarrow 9 = 7 + (2B+4)(1) \Rightarrow B = -1$

or

$\Rightarrow (A+B)x^2 + (A-B+C)x + (A-C) = 6x-3$

$A+B=0,\ A-B+C=6,\ A-C=-3.$

Then solve in any way we want.

$$\Rightarrow \frac{6x-3}{x^3-1} = \frac{1}{(x-1)} + \frac{-x+4}{(x^2+x+1)}$$

Ex. Find $\int \frac{4}{x^3+x^2-x-1}\,dx$

$$\begin{aligned} x^3+x^2-x-1 &= x^2(x+1)-(x+1) \\ &= (x^2-1)(x+1) \\ &= (x+1)^2(x-1) \end{aligned}$$

$$\Rightarrow \frac{4}{x^3+x^2-x-1} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$\Rightarrow 4 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$

let $x=1 \Rightarrow 4 = 4A \Rightarrow A = 1$

$x=-1 \Rightarrow 4 = -2C \Rightarrow C = -2$

$x=0 \Rightarrow 4 = 1 - B + 2 \Rightarrow B = -1$

$$\Rightarrow \int \frac{4}{x^3+x^2-x-1}\,dx = \int \frac{1}{x-1}\,dx - \int \frac{1}{x+1}\,dx - 2\int \frac{1}{(x+1)^2}\,dx$$

$$= \ln|x-1| - \ln|x+1| + \frac{2}{x+1} + C$$

Ex. How do we integrate $-\int \frac{x-4}{x^2+x+1}\,dx$? (from first example)

Ideal substitution: $u = x^2+x+1$ so $du = (2x+1)\,dx$

$$\begin{aligned} -\int \frac{x-4}{x^2+x+1}\,dx &= -\frac{1}{2}\int \frac{2x-8}{x^2+x+1}\,dx \\ &= -\frac{1}{2}\int \frac{2x+1-9}{x^2+x+1}\,dx \\ &= -\frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{9}{2}\int \frac{1}{x^2+x+1}\,dx \\ &= -\frac{1}{2}\ln(x^2+x+1) + \frac{9}{2}\int \frac{1}{x^2+x+1}\,dx \end{aligned}$$

Complete the square: $x^2+x+1 = (x^2+x+\frac{1}{4}) + 1 - \frac{1}{4}$
$= (x+\frac{1}{2})^2 + \frac{3}{4}$

So
$$\begin{aligned} \frac{9}{2}\int \frac{1}{x^2+x+1}\,dx &= \frac{9}{2}\int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx \\ &= \frac{9}{2}\int \frac{1}{\frac{3}{4}[\frac{4}{3}(x+\frac{1}{2})^2+1]}\,dx \\ &= \frac{9}{2}\cdot\frac{4}{3}\int \frac{1}{(\frac{2}{\sqrt{3}}(x+\frac{1}{2}))^2+1}\,dx \\ &= 6\,\frac{\sqrt{3}}{2}\int \frac{1}{u^2+1}\,du = 3\sqrt{3}\arctan u + C = 3\sqrt{3}\arctan\left(\tfrac{2}{\sqrt{3}}(x+\tfrac{1}{2})\right) + C. \end{aligned}$$

Let $u = \frac{2}{\sqrt{3}}(x+\frac{1}{2}) \Rightarrow du = \frac{2}{\sqrt{3}}\,dx$
$\Rightarrow dx = \frac{\sqrt{3}}{2}\,du.$

Thus $-\int \frac{x-4}{x^2+x+1}\,dx = -\frac{1}{2}\ln(x^2+x+1) + 3\sqrt{3}\arctan\left(\frac{2}{\sqrt{3}}(x+\frac{1}{2})\right) + C.$

Improper integrals

Note: We've only defined $\int_a^b f(x)\,dx$ when f is bounded on $[a,b]$.
What if f is unbounded at some point in $[a,b]$?
Can we integrate over unbounded intervals, e.g. on $[-\infty,\infty]$?

Def. Improper integral
Consider an interval $[a,b)$, where b is either a finite number or $b=\infty$.
Let f be defined on $[a,b)$ and integrable on each interval $[a,\delta]$ for $a<\delta<b$.
We define $\int_a^b f(x)\,dx = \lim_{\delta\to b^-}\int_a^\delta f(x)\,dx$, provided the limit exists as a finite number, ∞, or $-\infty$.
We can make a similar definition for f on $(a,b]$:
$$\int_a^b f(x)\,dx = \lim_{\delta\to a^+}\int_\delta^b f(x)\,dx.$$
If the limit is finite, we say the integral converges.
If the limit is $-\infty$ or ∞, we say it diverges to ∞ (or to $-\infty$).

Ex. $f(x)=\frac{1}{x}$
$$\begin{aligned}\int_0^1 \frac{1}{x}\,dx &= \lim_{\delta\to0^+} \ln|x|\Big|_\delta^1\\ &= \lim_{\delta\to0^+}(\ln|1|-\ln|\delta|)\\ &= \infty \qquad (\text{since } \ln|\delta|\to-\infty)\end{aligned}$$
$$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{\delta\to\infty}\int_1^\delta \frac{1}{x}\,dx\\ &= \lim_{\delta\to\infty}\ln|x|\Big|_1^\delta\\ &= \lim_{\delta\to\infty}(\ln|\delta|-\ln|1|)\\ &= \infty.\end{aligned}$$

Ex. $$\begin{aligned}\int_0^1\frac{1}{\sqrt{x}}\,dx &= \lim_{\delta\to0^+}\int_\delta^1 x^{-\frac12}\,dx\\ &= \lim_{\delta\to0^+} 2x^{\frac12}\Big|_\delta^1\\ &= \lim_{\delta\to0^+}(2-2\sqrt{\delta})\\ &= 2.\end{aligned}$$

Ex. $$\begin{aligned}\int_1^\infty \frac{1}{x^2}\,dx &= \lim_{\delta\to\infty}\int_1^\delta\frac{1}{x^2}\,dx\\ &= \lim_{\delta\to\infty} -\frac{1}{x}\Big|_1^\delta\\ &= \lim_{\delta\to\infty}\left(-\frac{1}{\delta}-(-1)\right)\\ &= 1.\end{aligned}$$

Ex. $\int_1^\infty \frac{1}{x^p}\,dx$ for $p>0$, $p\neq1$ fixed.
$$\begin{aligned}\Rightarrow \int_1^\infty\frac{1}{x^p}\,dx &= \lim_{\delta\to\infty}\int_1^\delta\frac{1}{x^p}\,dx\\ &= \lim_{\delta\to\infty}\frac{x^{1-p}}{1-p}\Big|_1^\delta\\ &= \lim_{\delta\to\infty}(\delta^{1-p}-1)\\ &= \begin{cases}\frac{1}{p-1} & \text{if } p>1\\ \infty & \text{if } 0<p\le1\end{cases}\end{aligned}$$
($p=1$ also diverges from before).

Ex. $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx = \int_{-\infty}^{0} \frac{1}{x^2+1}\,dx + \int_{0}^{\infty} \frac{1}{x^2+1}\,dx$ (provided these integrals exist)

$= \lim_{b\to-\infty} \int_{b}^{0} \frac{1}{x^2+1}\,dx + \lim_{d\to\infty} \int_{0}^{d} \frac{1}{x^2+1}\,dx$ (must let b, d go to $\pm\infty$ independently).

$= \lim_{b\to-\infty} \arctan x \Big|_{b}^{0} + \lim_{d\to\infty} \arctan x \Big|_{0}^{d}$

$= \lim_{b\to\infty} (0 - \arctan(b)) + \lim_{d\to\infty} (\arctan(d) - 0)$

$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$

$= \pi$