Ex. On a previous assignment (Assignment 3), we noted that the following was incorrect:

$$\int_{-1}^{1} \frac{1}{x^2} \not= -\frac{1}{x}\Big|_{-1}^{1} = -1 - (-1) = -2.$$

This is wrong since $\frac{1}{x}$ is not defined on $[-1,1]$ and is not bounded there.

Now we can say more:

$$\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx.$$

But
$$\int_{-1}^{0} \frac{1}{x^2}\,dx = \lim_{b\to 0^-} \int_{-1}^{b} \frac{1}{x^2}\,dx = \lim_{b\to 0^-} -\frac{1}{x}\Big|_{-1}^{b} = \lim_{b\to 0^-} -\frac{1}{b} - (-(-1)) = \infty$$

and
$$\int_{0}^{1} \frac{1}{x^2}\,dx = \lim_{b\to 0^+} \int_{b}^{1} \frac{1}{x^2}\,dx = \lim_{b\to 0^+} -\frac{1}{x}\Big|_{b}^{1} = \lim_{b\to 0^+} = \infty$$

Note. When we have the sum of two improper integrals, the limits must be evaluated independently.

For example, the following is incorrect:

$$\int_{-\infty}^{\infty} x\,dx \not= \lim_{d\to\infty} \int_{-d}^{d} x\,dx = \lim_{d\to\infty} \frac{1}{2}x^2\Big|_{-d}^{d} = \lim_{d\to\infty^+} \frac{1}{2}\left(d^2 - (-d)^2\right) = 0.$$
(future courses: Cauchy's principle value)

Instead, we have $\int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$, (where 0 is arbitrary)

and, for example,

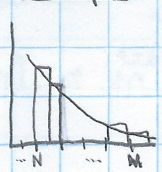
$$\int_{0}^{\infty} x\,dx = \lim_{d\to\infty} \frac{1}{2}x^2\Big|_{0}^{d} = \lim_{d\to\infty} \frac{1}{2}d^2 = \infty.$$

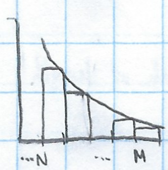
Thm. Integral convergence test for infinite series.

Let $N \in \mathbb{Z}$ and suppose $f(x)$ is continuous, decreasing, and nonnegative on $[N,\infty)$.

Then $\sum_{n=N}^{\infty} f(n)$ converges if and only if $\int_N^\infty f(x)\,dx$ converges.

Sketch of proof:





Let $M \in \mathbb{Z}$, $M > N$. From the pictures we can see that

$$\sum_{n=N}^{M} f(n) \geq \int_N^{M+1} f(x)\,dx \quad (1), \text{ and}$$
$$\sum_{n=N}^{M} f(n) \leq f(N) + \int_N^{M} f(x)\,dx. \quad (2)$$

Suppose $\int_N^\infty f(x)\,dx$ converges. Since f is nonnegative, $\int_N^M f(x)\,dx \leq \int_N^\infty f(x)\,dx$.

Also, $\sum_{n=N}^{M} f(n)$ is an increasing sequence. Thus (2) implies $\sum_{n=N}^{M} f(n)$ is bounded above.

Since they are increasing and bounded above, they converge as $M \to \infty$. Thus $\sum_{n=N}^{\infty} f(n)$ converges.

Now suppose $\int_N^\infty f(x)\,dx$ diverges. By (1), this implies that $\sum_{n=N}^{M} f(n)$ diverges, and hence $\sum_{n=N}^{\infty} f(n)$ diverges. $\square$

Ex. Consider $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p>0$.

Note that $f(x) = \frac{1}{x^p}$ is continuous, decreasing, and nonnegative on $[1, \infty)$.

Since $\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p>1$ and diverges for $p \le 1$ (last class), we have
$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$
converges for $p>1$ and diverges for $p \le 1$.

Note: If the improper integral converges, it does not tell us the sum of the series.

Ex. Determine if $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ and $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges.

We note that both $\frac{1}{x \ln x}$ and $\frac{1}{x(\ln x)^2}$ are continuous, decreasing, and nonnegative on $[2, \infty)$.

We have
$$\begin{aligned}\int_2^\infty \frac{1}{x \ln x} &= \lim_{d\to\infty} \int_2^d \frac{1}{x \ln x} \qquad u = \ln x,\ du = \frac{1}{x}\,dx\\ &= \lim_{d\to\infty} \int_{\ln 2}^{\ln d} \frac{1}{u}\,du\\ &= \lim_{d\to\infty} \ln|u| \Big|_{\ln 2}^{\ln d}\\ &= \lim_{d\to\infty} \left(\ln|\ln d| - \ln \ln 2\right)\\ &= \infty.\end{aligned}$$

Thus $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges.

Similarly,
$$\begin{aligned}\int_2^\infty \frac{1}{x(\ln x)^2}\,dx &= \lim_{d\to\infty} \int_2^d \frac{1}{x(\ln x)^2}\,dx\\ &= \lim_{d\to\infty} \int_{\ln 2}^{\ln d} \frac{1}{u^2}\,du\\ &= \lim_{d\to\infty} -\frac{1}{u}\Big|_{\ln 2}^{\ln d}\\ &= \lim_{d\to\infty} \left(-\frac{1}{\ln d} - \left(-\frac{1}{\ln 2}\right)\right)\\ &= \frac{1}{\ln 2}.\end{aligned}$$

Thus $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges (but not to $\frac{1}{\ln 2}$).

Applications of Integrals

Note: This is not course evaluated material. See slides on Brightspace.

Apl. Average value of a function

For f integrable on $[a,b]$, divide $[a,b]$ into n equal subintervals, each of length $\Delta x = \frac{b-a}{n}$. x_1 = first, x_2 = second, ...

Then average of these values is

Ex. Average value of $f(x) = x^2$ on $[0, 2]$

Ex. Average value of $f(x) = \sin x$ on

Apl. Volume

Area under the curve is $\int_a^b f(x)\,dx$ between a and b.

Then for a 3D shape over an axis x from a to b, define $A(x)$ as the surface area of the cross section at x.

Then the volume is Σ

Ex. Find volume formula of a sphere with radius r.

circle equation: $x^2+y^2=r^2$.

Rotate over x-axis.

$$V=\int_{-r}^{r} A(x)\,dx$$
$$=\int_{-r}^{r}$$
$$=$$
$$=$$

Apl. Arc length

with n sub intervals of width Δx, arc length is found by

$$\sqrt{\Delta x^2+(f(z_i)-f(z_{i-1}))^2}$$

By MVT and some cleanup, we get

$$\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

which gives us arc length of the curve between a and b.

Ex. Find arc length of $f(x)=x^2$ from 0 to 1.

$f'(x)=2x$, arc length $=\int_0^1 \sqrt{1+4x^2}\,dx$

trig sub: $x=\frac{1}{2}\tan\theta$

$$\Rightarrow \frac{1}{2}\int_0^{\arctan 2} \sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta = \frac{1}{2}\int_0^{\arctan 2} \sec^3\theta\,d\theta$$
$$=$$
$$=$$
$$=$$

Apl. Surface area

$$\int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx$$

Ex. Surface area of sphere

$$\int_{-r}^{r} 2\pi\sqrt{r^2-x^2}$$

Ex. Curve $f(x) = \frac{1}{x}$ rotated around x-axis for $x \geq 1$

$A(x) = \pi \left(\frac{1}{x}\right)^2$

Volume $= \int_1^\infty$

Surface area is

$$\int_1^\infty 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx = 2\pi \lim_{d\to\infty} \int_1^d \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx$$

For $x \geq 1$, $\frac{1}{x} > 0$ and $\sqrt{1+\frac{1}{x^4}} > 1$, thus

$$\int_1^d \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > \int_1^d \frac{1}{x}\,dx = \ln x \Big|_1^d = \ln d = \infty$$