Differential Equations

Def.

A differential equation is an equation which relates a function with its derivatives.
For example,

$y' = y$, or

$y' = -\frac{x}{y}$.

We have $y = y(x)$ as a function of x. Then we want to find all functions y that satisfy the equation.

Ex. What functions are proportional to their rate of change?
(solve $y' = ky$ for some constant k).

Integrating with respect to x won't help:

$\int y'\,dx = \int ky$, and how could we integrate what we want to find?

Note that $y(x) = 0$ is a solution.

For any other solution, we could divide by y (for at least some x):

$\frac{1}{y}y' = k$

Then integrate both sides with respect to x:

$$\int \frac{1}{y}y'\,dx = \int k\,dx$$
$$= kx + c_1$$

We can use substitution on left side:

$$\int \frac{1}{y}y'\,dx = \int \frac{1}{u}\,du \quad (\text{let } u = y,\ du = y'dx)$$
$$= \ln|y| + c_2$$

Thus $\ln|y| + c_2 = kx + c_1$

$$\Rightarrow \ln|y| = kx + c_3$$
$$\Rightarrow |y| = e^{kx+c_3}$$
$$\Rightarrow y = \pm e^{kx+c_3}$$
$$= \pm e^{kx}e^{c_3}$$
$$= ce^{kx}$$

(letting $c = \pm e^{c_3}$, thus $c \neq 0$ but we already showed that constant zero function is a solution, so c is arbitrary).

This worked since we could rearrange the diff. equation to the form

$$F(y)\,y' = G(x).$$

Integrating with respect to x yields

$$\int F(y)\,y'\,dx = \int G(x)\,dx$$
$$\Rightarrow \int F(y)\,dy = \int G(x)\,dx.$$

These diff. equations are called separable.

In Leibnitz notation:

$$\int F(y)\frac{dy}{dx}\,dx = \int F(y)\,dy$$

Ex. Solve $y' = \frac{-x}{y}$

$\Rightarrow yy' = -x$

$\Rightarrow \int yy'\,dx = \int y\,dy = \int -x\,dx$

$\Rightarrow \frac{1}{2}y^2 = -\frac{1}{2}x^2 + c_1$

$\Rightarrow x^2 + y^2 = c$ (letting $c = 2c_1$)

Which are circles. So slopes of tangent lines to circles at (x, y) satisfy $y' = -\frac{x}{y}$.

What curves are orthogonal (perpendicular) to the circles? (perpendicular $\Leftrightarrow$ slope is negative reciprocal)

Solve $y' = \frac{y}{x}$. (constant zero function is a solution, so for other y...)

$\Rightarrow \frac{1}{y}y' = \frac{1}{x}$

$\Rightarrow \int \frac{1}{y}y'\,dx = \int \frac{1}{y}\,dy = \int \frac{1}{x}\,dx$

$\Rightarrow \ln|y| = \ln|x| + c_1$

$\Rightarrow |y| = e^{\ln|x| + c_1}$

$= e^{\ln|x|}e^{c_1}$

$= |x|e^{c_1}$

$= c_2|x|$ (taking $c_2 = e^{c_1}$)

$\Rightarrow y = \pm c_2 x$

$= cx$ (taking $c = \pm c_2$)

Which are lines through the origin.

Thus we just showed that the orthogonal trajectories to the family of circles centred at the origin is the family of lines that pass through the origin.

Note. We will learn more about differential equations in MATH 2454.

Note. The remainder of this lecture is slides posted on Brightspace which covered supplementary material (applications of differential equations), specifically how to find the escape velocity.