Ex. Given a sequence $(a_n)_{n=0}^{\infty}$, the series
$$\sum_{n=0}^{\infty} a_n x^n$$
is called a power series (centred at 0) with coefficients a_n. x is a variable.
The series may converge for some values of x but not others.
Since $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ (for $x \neq 0$), but the right hand side is a_0 when $x=0$,
we use the convention that $0^0 = 1$ here (and only here).
Thus any power series (centred at 0) converges for $x=0$ to a_0.

Ex. $\sum_{n=1}^{\infty} n^n x^n$ (so $a_n = \begin{cases} 0 & \text{if } n=0 \\ n^n & \text{if } n \geq 1 \end{cases}$).
We fix x and use the root test.
$\lim_{n\to\infty} |n^n x^n|^{\frac{1}{n}} = \lim_{n\to\infty} n|x|$.
We need this ≤ 1 (otherwise, series diverges), which can only happen when $x=0$.
Thus $\sum_{n=1}^{\infty} n^n x^n$ converges only at $x=0$. (The coefficients grow "too fast.")

Ex. $\sum_{n=0}^{\infty} x^n$ is a power series ($a_n = 1$).
It is the geometric series – from MATH 1052 we proved this diverges for $|x| \geq 1$ and converges for $|x| < 1$ (to $\frac{1}{1-x}$).
Thus this series converges for x close enough to the centre.

Ex. $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$.
We fix x and use the ratio test.
$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)!} x^{n+1}}{\frac{1}{n!} x^n} \right| = \lim_{n\to\infty} \frac{n!}{(n+1)!} |x| = |x| \lim_{n\to\infty} \frac{1}{n+1} = |x|(0) = 0 < 1.$$
Thus for any fixed x, the series converges absolutely.

Note. It turns out that any power series centred at 0 must either:
- converge only at 0,
- converge for x in a bounded interval centred at 0, possibly converging or diverging at either endpoint,
- converges for all x.

Thm.
For the power series $\sum a_n x^n$, let
$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|,$$
provided the limit exists or is ∞.
Then the power series converges absolutely for $|x| < R$ and diverges for $|x| > R$.
R is called the radius of convergence.

Proof:
We use the ratio test on $\sum a_n x^n$.
We have $\lim_{n\to\infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = |x| \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. (*)

Case 1: $0 < R < \infty$.

Then thm. 9.6 implies $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R}$, and so limit ⊛ is $\frac{|x|}{R}$.

By the ratio test, the series converges absolutely if $\frac{|x|}{R} < 1 \Leftrightarrow |x| < R$, and diverges if $\frac{|x|}{R} > 1 \Leftrightarrow |x| > R$.

Case 2: $R = \infty$.

Then thm. 9.10 implies $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{\left|\frac{a_n}{a_{n+1}}\right|} = 0$.

Thus limit ⊛ $= |x|(0) = 0 < 1$ for any x.

The series converges for all x.

Case 3: $R = 0$.

Then thm. 9.10 implies $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1}{\left|\frac{a_{n+1}}{a_n}\right|} = \infty$.

Thus limit ⊛ $= \infty$ if $x \neq 0$.

The series diverges for $|x| > 0$. $\square$

Ex. $\sum_{n=1}^{\infty}\frac{1}{n}x^n$ $\left(a_n = \begin{cases} 0 & \text{if } n=0 \\ \frac{1}{n} & \text{if } n \geq 1\end{cases}\right)$

We have $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n}}{\frac{1}{n+1}}\right| = \lim_{n\to\infty}\frac{n+1}{n} = 1$.

Thus, the series converges

What about $x=1$, $x=-1$:

When $x=1$, the series is $\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges (harmonic series).

When $x=-1$, the series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, which converges conditionally.

Thus $\sum_{n=1}^{\infty}\frac{1}{n}x^n$ converges for x in $[-1, 1)$ (this is called the <u>interval of convergence</u>).

In general, the endpoints must be checked seperately.

Ex. We've seen:

<u>power series</u>	<u>interval of convergence</u>
$\sum x^n$	$(-1, 1)$
$\sum \frac{1}{n}x^n$	$[-1, 1)$
$\sum \frac{1}{n!}x^n$	$(-\infty, \infty)$.

Ex. $\sum_{n=1}^{\infty}\frac{1}{n^2}x^n$.

We have $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^2 = 1$.

This converges for $|x| < 1$.

At $x = 1$, series is $\frac{1}{n^2}$ which converges.

At $x = -1$, series is $\frac{(-1)^n}{n^2}$, which converges.

Interval of convergence is $[-1, 1]$.

Ex. $\sum_{n=0}^{\infty} 3^n x^{2n}$ $\quad (a_{2n} = 3^n,\ a_{2n+1} = 0)$

$\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ does not exist (every second coefficient is 0), so the theorem does not apply.

We use the ratio test on $\sum_{n=0}^{\infty} 3^n x^{2n}$.

$$\lim_{n\to\infty} \left|\frac{3^{n+1} x^{2(n+1)}}{3^n x^{2n}}\right| = \lim_{n\to\infty} 3x^2 = 3x^2.$$

Series converges if $3x^2 < 1 \Leftrightarrow x^2 < \frac{1}{3}$
$\Leftrightarrow |x| < \frac{1}{\sqrt{3}}$.

Series diverges if $3x^2 > 1 \Leftrightarrow x^2 > \frac{1}{3}$
$\Leftrightarrow |x| > \frac{1}{\sqrt{3}}$.

$R = \frac{1}{\sqrt{3}}$.

At $x = \frac{1}{\sqrt{3}}$, series is $\sum 3^n \left(\frac{1}{\sqrt{3}}\right)^{2n} = \sum \frac{3^n}{3^n} = \sum 1$ which diverges.

At $x = -\frac{1}{\sqrt{3}}$, series is $\sum 3^n \left(-\frac{1}{\sqrt{3}}\right)^{2n} = \sum \left(\frac{3^n}{3^n}\right) = \sum 1$ which diverges.

Interval of convergence is $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$.

<u>Alternative solution sketch:</u>

Let $y = x^2$. Then
$$\sum 3^n x^{2n} = \sum 3^n y^n.$$

We have $R = \lim_{n\to\infty} \left|\frac{3^n}{3^{n+1}}\right| = \frac{1}{3}$.

Then $\sum 3^n x^{2n}$ converges if $|y| < \frac{1}{3} \Rightarrow |x^2| < \frac{1}{3}$
$\Rightarrow |x| < \frac{1}{\sqrt{3}}$.

Ex. Power series can be centred at values other than 0.

The power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ centred at x_0 will still have a radius of convergence.

It will converge for $|x - x_0| < R$, (that is, for $x_0 - R < x < x_0 + R$)
and diverge for $|x - x_0| > R$.

Convergence at $x_0 \pm R$ still must be checked seperately.

We still have $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$ if it exists or is ∞.

Ex. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$ $\quad \left(\text{so } a_n = \begin{cases} 0 & \text{if } n = 0 \\ \frac{(-1)^{n+1}}{n} & \text{if } n \geq 1 \end{cases},\ \text{and } x_0 = 1\right)$

We have $\lim_{n\to\infty} \left|\frac{(-1)^{n+1}/n}{(-1)^{n+2}/(n+1)}\right| = \lim_{n\to\infty} \frac{n+1}{n} = 1$, so $R = 1$.

Converges for $|x - 1| < 1$, ie, for $0 < x < 2$.

At $x = 0$, series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n}{n} = -\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges.

At $x = 2$, series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, which converges conditionally.

The interval of convergence is $(0, 2]$.