Recall A power series converges to a function of x (for x in its interval of convergence).
We would like to know properties of the function:

- is it continuous?
- is it differentiable?
  - if so, can we differentiate term by term?
- is it integrable?
  - if so, can we integrate term by term?

Def.

The partial sums of a power series $\sum a_n x^n$ are polynomials and hence continuous.
If the power series converges, its sum is the limit of the sequence of partial sums.
Is the limit of a convergent sequence of continuous functions necessarily continuous? Next example: no.

Ex. Let $f_n(x) = x^n$ for $x \in [0,1]$. Each f_n is continuous on $[0,1]$.
For $0 \le x < 1$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$
For $x = 1$, $\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1^n = 1$
Thus the sequence of continuous functions $f_n(x) = x^n$ on $[0,1]$ converges to the discontinuous function $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$.
We essentially are fixing a value for x and looking at the sequence of numbers $f_n(x)$ seperately for each x. This is called pointwise converging.

Def.

The sequence f_n of functions on $S \subseteq \mathbb{R}$ converges pointwise to a function f defined on S if for all $x \in S$,
$$\lim_{n\to\infty} f_n(x) = f(x)$$
The ε, N definition of pointwise convergence:
$f_n \to f$ pointwise if for all $x \in S$ and for all $\varepsilon > 0$, there exists an N such that $n > N$ implies $|f_n(x) - f(x)| < \varepsilon$.
or $\forall x \in S, \forall \varepsilon > 0, \exists N, n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$.
Here, N may depend on ε and also on x. That is, we may choose N differently for different values of x.

Ex. In the previous example, given a fixed ε ($\varepsilon = 0.1$, say), increasingly large N are needed for x near 1.

At $x = \frac{1}{2}$, $x^n < 0.1$
$\Leftrightarrow \frac{1}{2^n} < 0.1$
$\Leftrightarrow 2^n > 10$
So for $n \ge 4$ can take $N = 4$.

At $x = \frac{9}{10}$, $x^n < 0.1$
$\Leftrightarrow \left(\frac{9}{10}\right)^n < 0.1$
$\Leftrightarrow \left(\frac{10}{9}\right)^n > 10$
$\Leftrightarrow n \ln \frac{10}{9} > \ln 10$
$\Leftrightarrow n > \frac{\ln 10}{\ln \frac{10}{9}} \approx 21.8\ldots$
can take $N = 22$.

At $x = \frac{99}{100}$, $x^n < 0.1$
$\Leftrightarrow \left(\frac{99}{100}\right)^n < 0.1$
$\Leftrightarrow n \ln \frac{99}{100} > \ln 10$
$\Leftrightarrow n > \frac{\ln 10}{\ln \frac{99}{100}} \approx 229.1$
can take $N = 230$.

It seems like one N will not work for all x in $[0,1]$ (which is okay for pointwise convergence).

What if we strengthen the definition such that one N must work for all x (N may still depend on ε). We would call this uniform convergence.

Def.

The sequence f_n of functions on $S \subseteq \mathbb{R}$ converges uniformly to a function f defined on S if
$\forall \varepsilon > 0, \exists N, \forall x \in S$ such that $n > N \Rightarrow |f_n(x) - f(x)| < \varepsilon$.

Note: If $f_n \to f$ is uniformly convergent, then $f_n \to f$ is pointwise convergent as well.

Ex. We saw $f_n(x) = x^n$ on $[0,1]$ converges pointwise to
$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

Does it converge uniformly to f on $[0,1]$?

Suppose it does. Then by definition, with $\varepsilon = \frac{1}{2}$ implies the existence of N such that for all $x \in [0,1]$,
$n > N \Rightarrow |x^n - f(x)| < \frac{1}{2}$.

Take $x \in [0,1)$ so $f(x) = 0$ and $n = N+1$. Then for all $x \in [0,1)$ we have
$x^{N+1} < \frac{1}{2}$.

Let $x_1 = \frac{1}{2^{\frac{1}{N+1}}}$. Then $x_1 > 0$. We have
$2 > 1 \Rightarrow 2^{\frac{1}{N+1}} > 1 \Rightarrow$

But $\frac{1}{2} > x_1^{N+1} = \left(\frac{1}{2^{\frac{1}{N+1}}}\right)^{N+1} = \frac{1}{2}$, which is a contradiction.

Thus f_n does not converge uniformly to f on $[0,1]$.

Uniform convergence is a stronger condition than pointwise convergence.

Note that if $f_n \to f$ converges uniformly on S, given $\varepsilon > 0$ there is an N such that
$|f_n(x) - f(x)| < \varepsilon$
$\Rightarrow -\varepsilon < f_n(x) - f(x) < \varepsilon$
$\Rightarrow f(x) - \varepsilon < f_n(x) < f(x) + \varepsilon$

That is, $f_n(x)$ lies in the strip between $f - \varepsilon$ and $f + \varepsilon$ when $n > N$.

Ex. Let $f_n(x) = \frac{1}{n}\sin(nx)$ for $x \in \mathbb{R}$.

Prove $f_n \to 0$ converges uniformly on $\mathbb{R}$. (constant zero function)

Rough: Let $\varepsilon > 0$. Then
$|f_n(x) - 0| = \frac{1}{n}|\sin(nx)| \le \frac{1}{n}$

We want $\frac{1}{n} < \varepsilon \Leftrightarrow n > \frac{1}{\varepsilon}$. Take $N = \frac{1}{\varepsilon}$.

Proof: Let $\varepsilon > 0$ be given and take $N = \frac{1}{\varepsilon}$. Then for any $x \in \mathbb{R}$, $n > N$ implies $n > \frac{1}{\varepsilon}$ and so $\frac{1}{n} < \varepsilon$.
Thus $|f_n(x) - 0| = \frac{1}{n}|\sin(nx)| \le \frac{1}{n} < \varepsilon$. $\square$