Thm 24.3

The uniform limit of a sequence of continuous functions is continuous.

Let f_n be a sequence of functions on $S \subseteq \mathbb{R}$ such that $f_n \to f$ uniformly on S, where f is a function on S.

If each f_n is continuous at $x_0 \in S$, then f is continuous at x_0.

Proof

Basic idea: we want $|f(x)-f(x_0)|$ small when x is near x_0. Since $f_n \to f$ uniformly, we can make $|f(x)-f_n(x)|$ and $|f_n(x_0)-f(x_0)|$ small for large enough n. Since each f_n is cont. at x_0, $|f_n(x)-f_n(x_0)|$ small for these large enough n, provided x is close enough to x_0.

We will use
$$\begin{aligned}|f(x)-f(x_0)| &= |f(x)-f_n(x)+f_n(x)-f_n(x_0)+f_n(x_0)-f(x_0)| \\ &\le |f(x)-f_n(x)|+|f_n(x)-f_n(x_0)|+|f_n(x_0)-f(x_0)|.\end{aligned}$$
for large enough n.

Let $\varepsilon>0$ be given. Since f_n converges uniformly to f, there is an N such that $x \in S$, including x_0, $n>N$ implies $|f_n(x)-f(x)|<\frac{\varepsilon}{3}$.

In particular, $|f_{N+1}(x)-f(x)|<\frac{\varepsilon}{3}$. $(*)$

Since f_{N+1} is continuous at x_0, there is $\delta>0$ such that $x \in S$ and $|x-x_0|<\delta$ imply
$|f_{N+1}(x)-f_{N+1}(x_0)|<\frac{\varepsilon}{3}$. $(*)(*)$

Thus $x \in S$ and $|x-x_0|<\delta$ imply
$$\begin{aligned}|f(x)-f(x_0)| &\le |f(x)-f_{N+1}(x)|+|f_{N+1}(x)-f_{N+1}(x_0)|+|f_{N+1}(x_0)-f(x_0)| \\ &< \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} \\ &= \varepsilon.\end{aligned}$$

Thus f is continuous at x_0. $\square$

Ex. Let $f_n(x)=(1-|x|)^n$ on $(-1,1)$.

Find the pointwise limit if it converges.

If $x=0$, then $1-|x|=1$, and so
$(1-|x|)^n = 1 \to 1$ as $n\to\infty$.

If $-1<x<1$, $x\neq 0$, then $0<|x|<1$, and so $0<1-|x|<1$. Thus $(1-|x|)^n \to 0$ as $n\to\infty$.

f_n converges pointwise to $f(x)=\begin{cases}0 & \text{if } -1<x<1,\ x\neq 0 \\ 1 & \text{if } x=0\end{cases}$.

Since each f_n is continuous on $(-1,1)$ but f is not, f_n does not converge to f uniformly on $(-1,1)$.

Ex. Consider $f_n(x)=x^n$ on $[0,1)$. (previously $[0,1]$)

Then the pointwise limit is $f(x)=0$ on $[0,1)$. This is continuous. But f_n does not converge uniformly to f on $[0,1)$. The previous proof that convergence is not uniform on $[0,1]$ will still work because $x=1$ was already excluded there.

The problem here is that x can act arbitrarily close to 1.

Ex. Let $0<b<1$ be fixed, and consider $f_n(x)=x^n$ on $[0,b]$.

Then $f_n \to 0$ uniformly on $[0,b]$.

Rough work: Let $\varepsilon>0$ be given.

Then we need N such that $n>N$ implies

$$|x^n - 0| < \varepsilon$$

for all $x \in [0,b]$.

We have $|x^n-0| = x^n \le b^n$ and $b^n<\varepsilon \Leftrightarrow n\ln b < \ln\varepsilon$

$\Leftrightarrow n > \frac{\ln\varepsilon}{\ln b}$ $(\ln b < \ln 1 = 0$, since $b<1)$

Choose $N=\frac{\ln\varepsilon}{\ln b}$.

Proof: Let $\varepsilon>0$ be given and take $N=\frac{\ln\varepsilon}{\ln b}$.

Then for any $x\in[0,b]$, $n>N$ implies

$$\begin{aligned} |x^n-0| = x^n \le b^n &\le b^N \\ &= b^{\frac{\ln\varepsilon}{\ln b}} \\ &= e^{\ln b^{\frac{\ln\varepsilon}{\ln b}}} \quad (\text{using } A=e^{\ln A}) \\ &= e^{\frac{\ln\varepsilon}{\ln b}\ln b} \\ &= e^{\ln\varepsilon} \\ &= \varepsilon. \quad \square \end{aligned}$$

In summary, let $f_n(x)=x^n$, $f(x)=\begin{cases}0 & \text{if } 0\le x<1\\ 1 & \text{if } x=1\end{cases}$, and let $0<b<1$ be fixed.

On $[0,1]$, $f_n\to f$ pointwise but not uniformly.

On $[0,1)$, $f_n\to 0$ pointwise but not uniformly.

On $[0,b]$, $f_n\to 0$ uniformly.

Thm. 25.2

Let (f_n) be a sequence of continuous functions on $[a,b]$ that converge uniformly to f on $[a,b]$. Then $\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.

Proof:

Theorem 24.3 implies f is continuous on $[a,b]$. Thus f_n-f is continuous and hence integrable on $[a,b]$ for each n.

Let $\varepsilon>0$ be given. Since $f_n\to f$ uniformly on $[a,b]$, there is an N such that for any x in $[a,b]$, $n>N$ implies $|f_n(x)-f(x)|<\frac{\varepsilon}{b-a}$. (*)

Thus $n>N$ implies

$$\begin{aligned} \left|\int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx\right| &= \left|\int_a^b (f_n(x)-f(x))\,dx\right| \\ &\le \int_a^b |f_n(x)-f(x)|\,dx \quad (\text{by thm. } 33.5) \\ &\le \int_a^b \frac{\varepsilon}{b-a}\,dx \quad (\text{by thm. } 33.4(i)) \\ &= \frac{\varepsilon}{b-a}\, b-a \\ &= \varepsilon \end{aligned}$$

Thus $\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$. $\square$

Ex. Consider $f_n(x) = x^n$ on $[0, b]$ for fixed $0 < b < 1$.

By thm. 25.2,

$$\lim_{n\to\infty} \int_0^b x^n \, dx = \int_0^b 0 \, dx = 0.$$

Check: $\lim_{n\to\infty} \int_0^b x^n \, dx = \lim_{n\to\infty} \frac{1}{n+1} x^{n+1} \Big|_0^b$

$= \lim_{n\to\infty} \frac{b^{n+1}}{n+1}$