Def.

Let (f_n) be a sequence of functions on $S \subseteq \mathbb{R}$.
The sequence is uniformly Cauchy on S if and only if
for any $\varepsilon > 0$, there is an N such that for all $x \in S$, $m,n > N$ implies $|f_n(x) - f_m(x)| < \varepsilon$.

Note. We want to show that uniformly Cauchy sequences are the same as uniformly convergent sequences.

Thm.

Suppose $f_n \to f$ uniformly on S.
Then f_n is uniformly Cauchy on S.

Proof:

Let $\varepsilon > 0$ be given. Then there is an N such that for all $x \in S$, $n > N$ implies $|f_n(x) - f(x)| < \frac{\varepsilon}{2}$.
Let $m,n > N$. Then

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\le |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &= |f_n(x) - f(x)| + |f_m(x) - f(x)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon, \text{ as required. } \square \end{aligned}$$

Thm 25.4

Let (f_n) be uniformly Cauchy on $S \subseteq \mathbb{R}$.
Then there is a function f on S such that $f_n \to f$ uniformly on S.

Proof

We must first find f. Since (f_n) is uniformly Cauchy, given $\varepsilon > 0$ there is an N such that
for all $x \in S$, $m,n > N$ implies $|f_n(x) - f_m(x)| < \varepsilon$.
Fix $x_0 \in S$. Then the above statement for the sequence of numbers $(f_n(x_0))$ is a Cauchy sequence.
Since Cauchy sequences converge, $\lim_{n\to\infty} f_n(x_0)$ exists.
Thus we define $f(x) = \lim_{n\to\infty} f_n(x)$ for each $x \in S$.
This implies $f_n \to f$ pointwise. We now prove $f_n \to f$ uniformly.
Let $\varepsilon > 0$ be given. Since (f_n) is uniformly Cauchy, there is N such that for all $x \in S$, $m,n > N$
implies $|f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$. $\star$
Let $m > N$ and $x \in S$. Then $\star$ implies

$$f_m(x) - \frac{\varepsilon}{2} < f_n(x) < f_m(x) + \frac{\varepsilon}{2}$$

for all $n > N$, and so

$$f_m(x) - \frac{\varepsilon}{2} \le f(x) \le f_m(x) + \frac{\varepsilon}{2} \quad (\text{since } \lim_{n\to\infty} f_n(x) = f(x)).$$

Thus $|f_m(x) - f(x)| \le \frac{\varepsilon}{2} < \varepsilon$ for all $x \in S$ and $m > N$.
Thus $f_n \to f$ uniformly on S. $\square$

Note: These theorems imply: f_n uniformly convergent on S $\Leftrightarrow$ f_n uniformly Cauchy on S.

Series of functions

Note: We say that the series of functions $\sum_{k=0}^{\infty} g_k(x)$ converges to $g(x)$ if and only if the sequence of partial sums converges to $g(x)$. If the sequence of partial sums converges uniformly to $g(x)$ on S, then we say the series converges uniformly to $g(x)$ on S. If the sequence of partial sums diverges to ∞ or $-\infty$, we say the series diverges to ∞ or $-\infty$. Otherwise $\sum_{k=0}^{\infty} g_k(x)$ is given no meaning.

A power series $\sum_{k=0}^{\infty} a_k x^k$ is a series of functions with $g_k(x) = a_k x^k$.
$\sum_{k=0}^{\infty} \frac{x^k}{1+x^k}$ is a series of functions that is not a power series (as it is written).

Thm. 25.5

Let $\sum_{k=0}^{\infty} g_k(x)$ be a series of functions on $S \subseteq \mathbb{R}$.
If each $g_k(x)$ is continuous on S and the series converges uniformly on S, then the sum $\sum_{k=0}^{\infty} g_k(x)$ is a continuous function on S.

Proof:
The partial sum $\sum_{k=0}^{n} g_k(x)$ is a finite sum of continuous functions and hence continuous.
Thus the sequence of partial sums is a uniformly convergent sequence of continuous functions on S.
Theorem 24.3 implies that the limit, which is the sum of the series, is continuous. $\square$

Ex. Let's write the Cauchy criterion for the sequence of partial sums of the series $\sum_{k=0}^{\infty} g_k(x)$.
$\sum_{k=0}^{\infty} g_k(x)$ is uniformly convergent on S if and only if
for any $\varepsilon > 0$, there is an N such that for all $x \in S$, $n, m > N$ implies

$|\sum_{k=0}^{m} g_k(x) - \sum_{k=0}^{n} g_k(x)| < \varepsilon$ (without loss of generality, take $n > m$)

$\Rightarrow |\sum_{k=0}^{n} g_k(x) - \sum_{k=0}^{m} g_k(x)| < \varepsilon$

$\Rightarrow |\sum_{k=m+1}^{n} g_k(x)| < \varepsilon$ (take $m_1 = m+1$, so $n \geq m_1$)

$\Rightarrow |\sum_{k=m_1}^{n} g_k(x)| < \varepsilon$.

i.e. for any $\varepsilon > 0$ there is N such that for all $x \in S$, $n \geq m > N$ implies $|\sum_{k=m}^{n} g_k(x)| < \varepsilon$.
This yields a useful result for uniform convergence.

Thm. 25.7: The Weierstrass M-test

Let (M_k) be a sequence of nonnegative numbers with $\sum M_k < \infty$.
If $|g_k(x)| \leq M_k$ for all $x \in S$, then $\sum g_k(x)$ converges uniformly on S.

Proof:
Let $\varepsilon > 0$ be given. Since $\sum M_k$ converges, the sequence of partial sums $\sum_{k=0}^{n} M_k$ is a Cauchy sequence of numbers. Thus there is an N such that $n \geq m > N$ implies
$\sum_{k=m}^{n} M_k < \varepsilon$.

Thus if $n \geq m > N$ and $x \in S$, we have

$$|\sum_{k=m}^{n} g_k(x)| \leq \sum_{k=m}^{n} |g_k(x)| \quad \text{(triangle inequality)}$$
$$\leq \sum_{k=m}^{n} M_k$$
$$< \varepsilon.$$

Thus $\sum g_k(x)$ is Cauchy on S, and hence converges uniformly on S. $\square$

Lemma

If $\sum g_k(x)$ converges uniformly on S, then
$\lim_{n\to\infty} \sup_{x\in S} \{ |g_n(x)| : x \in S \} = 0$.

Note: The lemma is useful for proving a series does not converge uniformly.

Proof: next lecture.