Lemma

If $\sum g_k(x)$ converges uniformly on S, then
$\lim_{x\to\infty} \sup_{x\in S} \{|g_n(x)| : x\in S\} = 0$

Proof:

Let $\varepsilon > 0$ be given. Since $\sum g_k(x)$ is Cauchy on S, there is an N such that for all $x \in S$,
$n \geq m > N$ implies $\left|\sum_{k=m}^{n} g_k(x)\right| < \frac{\varepsilon}{2}$.

Letting $m=n$, we have that for all $x\in S$, $n>N$ implies $|g_n(x)| < \frac{\varepsilon}{2}$.

Thus $n>N$ implies $\sup_{x\in S}\{|g_n(x)| : x\in S\} \leq \frac{\varepsilon}{2}$.

Since for any $\varepsilon>0$, there is N such that $n>N$ implies $0 \leq \sup_{x\in S}\{|g_n(x)| : x\in S\} \leq \frac{\varepsilon}{2} < \varepsilon$,
we have $\lim_{n\to\infty} \sup_{x\in S}\{|g_n(x)| : x\in S\} = 0$.

Ex. Consider the power series $\sum_{n=0}^{\infty} 3^{-n} x^n$.

$R = \lim_{n\to\infty}\left|\frac{3^{-n}}{3^{-(n+1)}}\right| = 3$.

At $x=\pm 3$, the series diverges.

The interval of pointwise convergence is $(-3,3)$.

Let $0<b<3$ be fixed. For $x\in[-b,b]$, we have
$|3^{-n}x^n| \leq 3^{-n} b^n = \left(\frac{b}{3}\right)^n$.

Since $\sum_{n=0}^{\infty}\left(\frac{b}{3}\right)^n$ converges (it is geometric with $\left|\frac{b}{3}\right|<1$), by the Weierstrass M-test,
$\sum_{n=0}^{\infty} 3^{-n}x^n$ converges uniformly on $[-b,b]$.

By thm. 25.5, the sum of the power series is continuous on $[-b,b]$.

Since this holds for any $b<3$, the sum is continuous on $(-3,3)$.

However, note that
$\sup_{x\in(-3,3)}\{|3^{-n}x^n| : x\in(-3,3)\} = 1$
for any n. Thus $\lim_{n\to\infty}\sup_{x\in(-3,3)}\{|3^{-n}x^n| : x\in(-3,3)\} = 1 \neq 0$, and so by the previous lemma, it does not converge uniformly on $(-3,3)$.

In summary, $\sum_{n=0}^{\infty} 3^{-n}x^n$ converges pointwise on $(-3,3)$ to a continuous function.

It converges uniformly on $[-b,b]$ for any $0<b<3$, but does not converge uniformly on $(-3,3)$.

In fact, we happen to know $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$. Thus
$\sum_{n=0}^{\infty} 3^{-n}x^n = \sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \frac{1}{1-\frac{x}{3}} = \frac{3}{3-x}$ for $\left|\frac{x}{3}\right|<1$, i.e. for $|x|<3$, i.e. for $-3<x<3$.

We can verify that $\frac{3}{3-x}$ is continuous on $(-3,3)$.

The example is still useful as it will generalize and can be applied to series where we don't know the sum.

Thm. 26.1 (and Corollary 26.2)

Let $\sum a_n x^n$ be a power series with $R>0$ (possibly $R=\infty$).

If $0<R_1<R$, then the power series converges uniformly on $[-R_1,R_1]$ and converges to a continuous function on $(-R_1,R_1)$.

Proof:

The series $\sum a_n x^n$ and $\sum |a_n| x^n$ have the same radius of convergence (by ratio test).

Since $|R_1| < R$, R_1 is in the interval of convergence for $\sum |a_n| x^n$, and so $\sum |a_n| R_1^n < \infty$.

Since for any $x \in [-R_1, R_1]$ we have $|a_n x^n| \le |a_n| R_1^n$, the Weierstrass M-test implies $\sum a_n x^n$ converges uniformly on $[-R_1, R_1]$.

By thm. 25.5, the limit function is continuous on $[-R_1, R_1]$.

If $x_0 \in (-R, R)$, then $x_0 \in (-R_1, R_1)$ for some $R_1 < R$.

Thus the above implies the limit function is continuous at x_0. $\square$

Note: A power series may or may not converge uniformly on its full interval of convergence.

Lemma 26.3

If $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, then

$\sum_{n=0}^{\infty} n a_n x^{n-1}$ ("derivative"), and

$\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ ("integral")

also have radius of convergence R.

Note: The interval of convergence may change.

Proof:

Note that $\sum n a_n x^{-1}$ and $\sum n a_n x^n$ have the same radius of convergence, as do $\sum \frac{a_n}{n+1} x^{n+1}$ and $\sum \frac{a_n}{n+1} x^n$ (since multiplying/dividing by x will not change convergence).

We have $\lim_{n\to\infty} \left|\frac{n a_n}{(n+1) a_{n+1}}\right| = \left(\lim_{n\to\infty} \frac{n}{n+1}\right)\left(\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|\right)$

$= (1)(R)$

and $\lim_{n\to\infty} \left|\frac{a_n/(n+1)}{a_{n+1}/(n+2)}\right| = \left(\lim_{n\to\infty} \frac{n+2}{n+1}\right)\left(\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|\right)$

$= (1)(R)$. $\square$

Thm. 26.4

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then

$\int_0^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ for $|x| < R$.

Proof:

Fix x with $|x| < R$. We prove the case when $x > 0$ ($x < 0$ is similar & in the textbook).

By thm. 26.1, $\sum_{n=0}^{\infty} a_n t^n$ converges uniformly to $f(t)$ on $[0, x]$, and so the sequence of partial sums $\sum_{k=0}^{n} a_k t^k$ with $t \in [0, x]$ converges uniformly to $f(t)$. Thus

$$
\begin{aligned}
\int_0^x f(t)\,dt &= \lim_{n\to\infty} \int_0^x \left(\sum_{k=0}^{n} a_k t^k\right) dt && \text{(by thm. 25.2)} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \int_0^x t^k\,dt && \text{(by thm. 33.3, since the sum is finite)} \\
&= \lim_{n\to\infty} \sum_{k=0}^{n} a_k \frac{x^{k+1} - 0}{k+1} \\
&= \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1}. \quad \square
\end{aligned}
$$

Ex. $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x|<1$.

Replace x with t and integrate from 0 to x.

$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \int_0^x \frac{1}{1-t}\,dt$$
$$= -\ln|1-t|\Big|_0^x$$
$$= -\ln|1-x|$$
$$= -\ln(1-x) \qquad (\text{since } |x|<1)$$

Thus $-\ln(1-x) = \sum_{n=0}^{\infty} \frac{1}{n} x^n$ for $|x|<1$.

ie. $x=\frac{1}{2}$, $\sum \frac{1}{n2^n} = -\ln(1-\frac{1}{2}) = \ln 2$.