**Thm. 26.5**

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$.
Then $f$ is differentiable on $(-R, R)$ with $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for $|x| < R$.

**Proof:**

Consider $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$, which will converge for $|x| < R$.
For $R_1$ with $0 < R_1 < R$, we have

$$\int_{-R_1}^{x} g(t)\,dt = \int_{-R_1}^{0} g(t)\,dt + \int_0^x g(t)\,dt$$
$$= \int_{-R_1}^{0} g(t)\,dt + f(x) - a_0,$$

so $f(x) = \int_{-R_1}^{x} g(t)\,dt + k$ (where $k = a_0 - \int_{-R_1}^{0} g(t)\,dt$ is a constant).
By thm 26.1, $g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ is continuous on $(-R, R)$ and so FTC II implies $f$ is differentiable on $(-R_1, R_1)$ with $f'(x) = g(x)$.
Since $R_1 < R$ is arbitrary, we have $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for $|x| < R$. □

**Thm. 26.6 Abel's Theorem**

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R < \infty$.
If the series converges at $x = R$, then $f$ is continuous at $x = R$.
If the series converges at $x = -R$, then $f$ is continuous at $x = -R$.
**Proof:** omitted, but available in the textbook.

**Ex.** We saw $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$ (geometric series).
Differentiating: $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$ for $|x| < 1$. ✪
Again: $\sum_{n=2}^{\infty} n(n-1) x^{n-2} = \frac{2}{(1-x)^3}$ for $|x| < 1$.
$\Rightarrow \sum_{n=0}^{\infty} (n+2)(n+1) x^n = \frac{2}{(1-x)^3}$ (reindexing).
We can multiply both sides of ✪ by $x$:
$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2} \text{ for } |x| < 1.$$
Let $x = \frac{1}{2}$, for example:
$$\sum_{n=1}^{\infty} \frac{n}{2^n} = \frac{1/2}{(1-\frac{1}{2})^2} = 2.$$
Integrating the geometric series $\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}$ term by term:
$$\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = -\ln(1-t)\Big|_0^x$$
$$= -\ln(1-x) \text{ for } |x| < 1.$$
Hence $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{1}{n} x^n$ ($R$ is still 1, but now the series converges at $-1$. Interval of convergence is $[-1, 1)$.)
By Abel's theorem, $-\sum_{n=1}^{\infty} \frac{x^n}{n}$ is continuous at $x = -1$, as is $\ln(x-1)$. Thus they must be equal at $x = -1$.
That is, $\sum \frac{(-1)^{n+1}}{n} = \ln 2$.

**Ex.** Consider the power series $f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$. Interval of convergence is $(-\infty, \infty)$.

$f'(x) = \sum_{n=1}^{\infty} \frac{n}{n!} x^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = f(x)$.

Let $y = f(x)$. We have $y' = y$. (differential equation).

If $y$ is not the constant zero function:

$$\frac{1}{y} y' = 1 \Rightarrow \int \frac{1}{y} dy = \int 1 \, dx$$
$$\Rightarrow \ln|y| = x + c_1$$
$$\Rightarrow |y| = e^{x+c_1} = e^{c_1} e^x$$
$$\Rightarrow y = \pm e^{c_1} e^x = ce^x.$$

Thus $f(x) = ce^x$ for some $c$.

Letting $x = 0$ yields $\frac{1}{0!} = ce^0 \to c = 1$.

Hence $\sum_{n=0}^{\infty} \frac{1}{n!} x^n = e^x$ for $x \in \mathbb{R}$.

For example, $\sum_{n=0}^{\infty} \frac{1}{n!} (-x^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}$ (still a power series.)

Integrating, we get

$$\int_0^x e^{-t^2} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} = x - \frac{1}{3}x^3 + \frac{1}{10}x^5 + \dots$$ (by FTCII. This is an antiderivative of $e^{-x^2}$ as a power series.)

**Ex.** Consider the Fibonacci sequence:

$F_1 = 1$, $F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. $(1, 1, 2, 3, 5, 8, 13, 21, \dots)$

We will form the power series with coefficients $F_n$ (the generating series for $F_n$).

$$\begin{aligned} g(x) &= \sum_{n=1}^{\infty} F_n x^n \\ &= F_1 x + F_2 x^2 + \sum_{n=3}^{\infty} (F_{n-1} + F_{n-2}) x^n \\ &= x + x^2 + x \sum_{n=3}^{\infty} F_{n-1} x^{n-1} + x^2 \sum_{n=3}^{\infty} F_{n-2} x^{n-2} \\ &= x + x^2 + x \sum_{n=2}^{\infty} F_n x^n + x^2 \sum_{n=1}^{\infty} F_n x^n \\ &= x + x \sum_{n=1}^{\infty} F_n x^n + x^2 \sum_{n=1}^{\infty} F_n x^n \\ &= x + x g(x) + x^2 g(x). \end{aligned}$$

Thus $g(x) = x + xg(x) + x^2 g(x)$.

$\Rightarrow g(x)(1 - x - x^2) = x$

$\Rightarrow g(x) = \frac{x}{1-x-x^2} = -\frac{x}{x^2+x-1}$.

Note that $x^2 + x - 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{5}}{2} = -\frac{1 \mp \sqrt{5}}{2}$.

Let $r_1 = \frac{1+\sqrt{5}}{2}$ and $r_2 = \frac{1-\sqrt{5}}{2}$. Then $x^2 + x - 1 = 0 \Rightarrow x = -r_1, -r_2$.

Then $x^2 + x - 1 = (x + r_1)(x + r_2) \Rightarrow g(x) = \frac{-x}{(x+r_1)(x+r_2)} = \frac{A}{x+r_1} + \frac{B}{x+r_2} \Rightarrow -x = A(x+r_2) + B(x+r_1)$. (partial fractions).

$x = -r_1 \Rightarrow r_1 = A(r_2 - r_1) = A(-\sqrt{5}) \Rightarrow A = -\frac{r_1}{\sqrt{5}}$.

$x = -r_2 \Rightarrow r_2 = B(r_1 - r_2) = B\sqrt{5} \Rightarrow B = \frac{r_2}{\sqrt{5}}$.

$$\begin{aligned} \text{Hence } g(x) &= \frac{1}{\sqrt{5}}\left[\frac{r_2}{x+r_2} - \frac{r_1}{x+r_1}\right] = \frac{1}{\sqrt{5}}\left[\frac{1}{1+\frac{x}{r_2}} - \frac{1}{1+\frac{x}{r_1}}\right] \\ &= \frac{1}{\sqrt{5}}\left[\frac{1}{1-r_1x} - \frac{1}{1-r_2x}\right] \\ &= \frac{1}{\sqrt{5}}\left[\sum_{n=0}^{\infty}(r_1x)^n - \sum_{n=0}^{\infty}(r_2x)^n\right] \\ &= \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} (r_1^n - r_2^n) x^n. \end{aligned}$$

(note $r_1 r_2 = \frac{1-5}{4} = -1$, so $\frac{1}{r_1} = -r_2$, $\frac{1}{r_2} = r_1$.)

(but $\sum x^n = \frac{1}{1-x}$ for $|x|<1$, so $\sum (rx)^n = \frac{1}{1-rx}$ for $|rx|<1 \Rightarrow |x| < \frac{1}{r}$.)

But $g(x) = \sum_{n=1}^{\infty} F_n x^n$, and so

$$F_n = \frac{1}{\sqrt{5}}(r_1^n - r_2^n) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

is an explicit formula for the Fibonacci numbers.