Note: Previous results for power series centred at 0 also hold if centred elsewhere, ie. term by term integration, etc.

Ex. Suppose we have a power series centred at c.

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \ldots$$
$$f'(x) = \sum_{n=1}^{\infty} a_n n(x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \ldots$$
$$f''(x) = \sum_{n=2}^{\infty} a_n n(n-1)(x-c)^{n-2} = 2a_2 + 2\cdot 3\, a_3(x-c) + 3\cdot 4\, a_4(x-c)^2 + \ldots$$
$$f'''(x) = \sum_{n=3}^{\infty} a_n n(n-1)(n-2)(x-c)^{n-3} = 2\cdot 3 a_3 + 2\cdot 3\cdot 4\, a_4(x-c) + 3\cdot 4\cdot 5\, a_5(x-c)^2 + \ldots$$
$$f^{(4)}(x) = \sum_{n=4}^{\infty} a_n n(n-1)(n-2)(x-c)^{n-4} = 2\cdot 3\cdot 4 + 2\cdot 3\cdot 4\cdot 5\, a_5(x-c) + 3\cdot 4\cdot 5\cdot 6\, a_6(x-c)^2 + \ldots$$

Let $x = c$.

$$f'(c) = a_1$$
$$f''(c) = 2a_2$$
$$f'''(c) = 2\cdot 3\, a_3$$
$$f^{(4)}(c) = 2\cdot 3\cdot 4\, a_4$$

Continuing in this manner, we see that
$$a_n = \frac{f^{(n)}(c)}{n!} \quad \text{(can be proved with induction).}$$

Thus, starting with a power series $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$, we have a formula for its coefficients.

If we start with a function (not necessarily a power series) and form the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ with coefficients $a_n = \frac{f^{(n)}(c)}{n!}$, it will often converge to f (though not always.)

Def.

Let f be defined on an open interval containing c, and suppose that all order of derivatives of f exist at c.

Then
$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$
is the Taylor series for f centred at c.

A Taylor series centred at 0 is often called a MacLaurin series.

For $n \geq 1$, the remainder is
$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Thus $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \iff \lim_{n\to\infty} R_n(x) = 0.$

Note: as we will eventually see, there are functions that are not equal to their Taylor series on any open interval containing c. Thus, some care needs to be taken here.

Ex. Let's find the Taylor series for $f(x) = \sin x$ centred at 0.

$$\begin{aligned} f(x) &= \sin x & f(0) &= 0 \\ f'(x) &= \cos x & f'(0) &= 1 \\ f''(x) &= -\sin x & f''(0) &= 0 \\ f'''(x) &= -\cos x & f'''(0) &= -1 \end{aligned}$$

$$\Rightarrow f^{(k)}(0) = \begin{cases} 0 & \text{if } k \text{ is even} \\ 1 & \text{if } k = 4m+1,\ m \in \mathbb{Z} \\ -1 & \text{if } k = 4m+3,\ m \in \mathbb{Z} \end{cases}$$

The Taylor series for $\sin x$ centred at 0 is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!} x^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

We can find the radius of convergence, with the ratio test.

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(2n+3)!} \frac{(2n+1)!}{(-1)^n} \frac{x^{2n+3}}{x^{2n+1}} \right| = x^2 \lim_{n\to\infty} \frac{(2n+1)!}{(2n+3)!}$$
$$= x^2 \lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)} = 0 < 1$$

for any given x.

Thus it converges for all x and $R = \infty$.

We don't yet know if it converges to $\sin x$.

Ex. Find the Taylor series for $f(x) = \ln x$ centred at 1.

$$\begin{aligned} f(x) &= \ln x \\ f'(x) &= \frac{1}{x} = x^{-1} \\ f''(x) &= -x^{-2} \\ f'''(x) &= 2x^{-3} \\ f^{(4)}(x) &= -2\cdot 3x^{-4} \\ f^{(5)}(x) &= 2\cdot 3\cdot 4x^{-5} \end{aligned}$$

Note that $f(1) = \ln 1 = 0$.

For $k \geq 1$,

$$f^{(k)}(x) = (-1)^{k+1}(k-1)!\, x^{-k},$$

and so $f^{(k)}(1) = (-1)^{k+1}(k-1)!$

$$\frac{f^{(k)}(1)}{k!} = \frac{(-1)^{k+1}(k-1)!}{k!} = \frac{(-1)^{k+1}}{k}$$

The Taylor series for $\ln x$ centred at $x=1$ is

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k$$

$$R = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{n} \frac{n+1}{(-1)^{n+2}} \right| = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

It converges for $|x-1| < 1$.

At $x=0$, series is $\sum \frac{(-1)^{k+1}}{k}(-1)^k = -\sum \frac{1}{k}$, which diverges.

At $x=2$, series is $\sum \frac{(-1)^{k+1}}{k}$, which converges.

Interval of convergence is $(0,2]$.

We check if the series converges to $\ln x$ for $(0,2]$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(x-1)^k = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}(-1)^k}{k}(1-x)^k$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{2k+1}}{k}(1-x)^k \qquad \text{Let } u = 1-x, \text{ so } 0 < x \leq 2 \Leftrightarrow -1 \leq u < 1$$
$$= -\sum_{k=1}^{\infty} \frac{u^k}{k}$$
$$= \ln(1-u) \qquad \text{(by previous class)}$$
$$= \ln x$$

Note: We know that if a function f is equal to a power series $\sum a_n (x-c)^n$ centred at c, then we saw $a_n = \frac{f^{(n)}(c)}{n!}$, and so that power series is the Taylor series for f centred at c.

Ex. We saw $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ on $\mathbb{R}$, so the Taylor series for e^x centred at 0 is $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, and it converges to e^x.

Ex. Find the Taylor series for e^{-x^2} centred at 0, and for e^x centred at 2.

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}.$$

Since $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n!} = e^{x-2} = e^x e^{-2}$, $e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!}(x-2)^n$.

Note: Suppose $\sum_{n=0}^{\infty} a_n (x-c)^n = f(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$ for $|x-c| < R$.

Then $a_n = \frac{f^{(n)}(c)}{n!} = b_n$.

Thus if two power series converge to the same function, then their coefficients match.