The first 30 minutes of the lecture were used on the course outline.

Integration

We would like to find the "signed" area under a curve (anything above the x-axis adds to the area, and anything below the x-axis removes from the area).

Def.

Let f be a bounded function on $[a,b]$. Letting $S \subseteq [a,b]$, we let

$M(f,S) = \sup\{f(x) : x \in S\}$ — sup is the lowest upper bound.

$m(f,S) = \inf\{f(x) : x \in S\}$ — inf is the highest lower bound.

A partition of $[a,b]$ is a finite sequence (t_n) with $t_0 = a$, $t_n = b$, and $t_0 < t_1 < \dots < t_n$.

The upper Darboux sum $U(f,P)$ of f with respect to P is $U(f,P)$ is

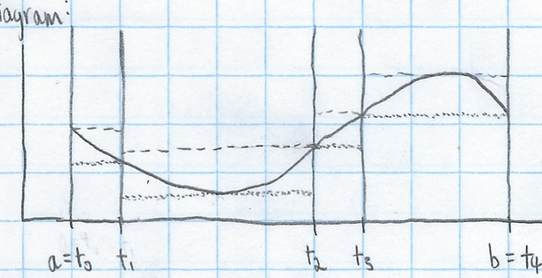
$$U(f,P) = \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}),$$

and the lower Darboux sum $L(f,P)$ is

$$L(f,P) = \sum_{k=1}^{n} m(f,[t_{k-1},t_k])(t_k - t_{k-1}).$$

Side note: if we want "unsigned" area, we can just take the absolute value of the function.

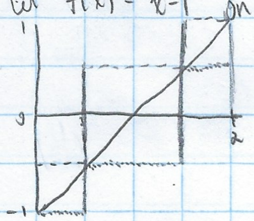
Diagram:



Dashed lines represent the upper Darboux sums.
Dotted lines represent the lower Darboux sums.
note: area is the rectangle formed from the lines to the x-axis. rectangles below the x-axis are counted with negative weight.

$U(f,P)$ overestimates the area under the curve, while $L(f,P)$ underestimates the area.

Ex. Let $f(x) = x-1$ on $[0,2]$ with partition $t_0 = 0$, $t_1 = \frac{1}{2}$, $t_2 = \frac{3}{2}$, $t_3 = 2$.



$$\begin{aligned} U(f,P) &= \sum_{k=1}^{3} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) \\ &= M(f,[0,\tfrac{1}{2}])(\tfrac{1}{2}) + M(f,[\tfrac{1}{2},\tfrac{3}{2}])(1) + M(f,[\tfrac{3}{2},2])(\tfrac{1}{2}) \\ &= (-\tfrac{1}{2})(\tfrac{1}{2}) + (\tfrac{1}{2})(1) + (1)(\tfrac{1}{2}) \\ &= \tfrac{3}{4}. \end{aligned}$$

Ex. Let $f(x) = x-1$ on $[0,2]$ with partition $t_0 = 0$, $t_1 = \frac{3}{2}$, $t_2 = \frac{7}{4}$, $t_3 = 2$.

$$U(f,P) = (\tfrac{1}{2})(\tfrac{3}{2}) + (\tfrac{3}{4})(\tfrac{1}{4}) + (1)(\tfrac{1}{4}) = \tfrac{19}{16}.$$

Note. We note that

$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n} M(f,[t_{k-1},t_k])(t_k - t_{k-1}) \\ &\le \sum_{k=1}^{n} M(f,[a,b])(t_k - t_{k-1}) \\ &= M(f,[a,b]) \cdot \sum_{k=1}^{n} (t_k - t_{k-1}) \\ &= M(f,[a,b])(\cancel{t_1} - t_0 + \cancel{t_2} - \cancel{t_1} + \dots + t_k - \cancel{t_{k-1}}) \\ &= M(f,[a,b])(b-a). \end{aligned}$$

which is a telescoping series.

Similarily, $L(f,P) \ge m(f,[a,b])(b-a)$.

Thus $m(f,[a,b])(b-a) \le L(f,P) \le U(f,P) \le M(f,[a,b])(b-a)$. ⊛

(since inf ≤ sup)

Def.

The upper Darboux integral $U(f)$ of f over $[a,b]$ is
$$U(f) = \inf\{U(f,P) : P \text{ is a partition of } [a,b]\}$$
and the lower Darboux integral $L(f)$ is
$$L(f) = \sup\{L(f,P) : P \text{ is a partition of } [a,b]\}.$$
By ⊛, we are taking the the sup and inf of bounded sets, and so $U(f)$ and $L(f)$ are real numbers.