Thm. 31.3 Taylor's Theorem

Let f be defined on (a,b) with $a<c<b$ (a can be finite or $-\infty$, b can be finite or ∞.)
Suppose the n-th derivative $f^{(n)}(x)$ exists on (a,b).
Then for each $x \in (a,b)$, $x \neq c$, there exists some y between c and x such that
$$R_n(x) = \frac{F^{(n)}(y)}{n!}(x-c)^n.$$

Proof:

Fix $x \neq c$ and $n \in \mathbb{N}$. Then there is a unique number M satisfying
$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{M(x-c)^n}{n!}. \quad \star$$
By definition of $R_n(x)$, this implies
$$R_n(x) = \frac{M(x-c)^n}{n!}.$$
We must show that $M = f^{(n)}(y)$ for some y between x and c.
Consider $g(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!}(t-c)^k + \frac{M(t-c)^n}{n!} - f(t)$.
Then $g(c) = \frac{f(c)}{0!} - f(c) = 0$.
By $\star$, $f(x) = 0$. Also,
$$g'(t) = \sum_{k=1}^{n-1} \frac{f^{(k)}(c)}{(k-1)!}(t-c)^{k-1} + \frac{M(t-c)^{n-1}}{(n-1)!} - f'(t).$$
Then $g'(c) = \frac{f'(c)}{0!} - f'(c) = 0$.
$$\Rightarrow g''(t) = \sum_{k=2}^{n-1} \frac{f^{(k)}(c)}{(k-2)!}(t-c)^{k-2} + \frac{M(t-c)^{n-2}}{(n-2)!} - f''(t)$$
$$\Rightarrow g''(c) = \frac{f''(c)}{0!} - f''(c) = 0.$$
Continuing in this manner, we have
$$g^{(n-1)}(t) = \sum_{k=n-1}^{n-1} \frac{f^{(k)}(c)}{(k-(n-1))!}(t-c)^{k-(n-1)} + \frac{M(t-c)^{n-(n-1)}}{(n-(n-1))!} - f^{(n-1)}(t)$$
$$= \frac{f^{(n-1)}(c)}{0!} + \frac{M(t-c)}{1!} - f^{(n-1)}(t)$$
$$\Rightarrow g^{(n-1)}(c) = f^{(n-1)}(c) - f^{(n-1)}(c) = 0.$$
Thus $g(x) = 0$ and $g^{(k)}(c) = 0$ for $0 \le k \le n-1$.
$g(t)$ is differentiable on (a,b). (since f is differentiable.)
Since $g(c) = 0 = g(x)$, so by Rolle's theorem (or MVT) there is some x_1 between x and c with $g'(x_1) = 0$.
Since $g'(x_1) = 0 = g'(c)$, by Rolle's theorem there is x_2 between x_1 and c with $g''(x_2) = 0$.
Repeating this argument, we get $g'''(x_3) = 0$ for x_3 between x_2 and c.
This process continues until we have x_n between c and x_{n-1} with $g^{(n)}(x_n) = 0$.
But $g^{(n)}(t) = M - f^{(n)}(t)$, and so
$$M - f^{(n)}(x_n) = 0 \Rightarrow f^{(n)}(x_n) = M.$$
Thus letting $y = x_n$ completes the proof. □

Lemma:

Let $b>0$ be constant with respect to n. Then
$$\lim_{n\to\infty} \frac{b^n}{n!} = 0.$$

Proof:

Let $c \in \mathbb{Z}$ with $c \ge b$. For $n > c$, we have
$$0 < \frac{b^n}{n!} \le \frac{c^n}{n!} = \underbrace{\left(\frac{c}{1}\right)\left(\frac{c}{2}\right)\cdots\left(\frac{c}{c-1}\right)}_{\text{each is } \le c.}\underbrace{\left(\frac{c}{c}\right)\left(\frac{c}{c+1}\right)\cdots\left(\frac{c}{n-1}\right)\left(\frac{c}{n}\right)}_{\text{each is } \le 1.}$$

$\Rightarrow \leq (c^{c-1})\left(\frac{c}{n}\right) = \frac{c^c}{n}$. $\left(\frac{c^c}{n} \to 0 \text{ as } n \to \infty\right)$.

By squeeze theorem, $\lim_{n\to\infty} \frac{b^n}{n!} = 0$.

Corollary 31.4

Let f be defined on (a,b) with $a<c<b$.

If all derivatives $f^{(n)}(x)$ exist on (a,b) and all are bounded by a single constant B (ie. $|f_n(x)| \leq B$), then $\lim_{n\to\infty} R_n(x) = 0 \quad \forall x \in (a,b)$.

Proof:

Let $x \in (a,b)$. Theorem 31.3 implies

$|R_n(x)| \leq \frac{B}{n!}|x-c|^n$ for all n.

Since $\lim_{n\to\infty} \frac{|x-c|^n}{n!} = 0$ (by previous lemma), $R_n(x) \to 0$ as $n \to \infty$. □

Ex. Recall the Taylor series for $\sin x$, which converges for all x.

Since all derivatives of $\sin x$ (namely $\pm \sin x$, $\pm \cos x$) are bounded by 1 on $\mathbb{R}$, by Corollary 31.4 we have $R_n(x) \to 0$ for $x \in \mathbb{R}$.

The Taylor series for $\sin x$ converges to $\sin x$ for all $x \in \mathbb{R}$.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

Ex. Let's find the Taylor series for $\cos x$ centred at 0.

We could use the same technique used for $\sin x$, or we can differentiate $\sin x$ term by term (by thm. 26.5).

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \quad (\text{for all } x).$$

Ex. Using Corollary 31.4 to show the Taylor series for e^x converges to e^x for all x (which we know by other means), centred at 0.

Taylor series centred at 0 is $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$.

On $(-b,b)$, $|f^{(n)}(x)| = |e^x| \leq e^b$.

By Corollary 31.4, the Taylor series converges to e^x on $(-b,b)$.

Since b is arbitrary, the Taylor series converges to e^x for all $x \in \mathbb{R}$.

(In particular, given x we can take b with $x \in (-b,b)$.)

Application:

Taylor series can be used for approximation.

For example, $e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$

$= \sum_{k=0}^{n} \frac{1}{k!} x^k + R_n(x)$

for $R_n(x) = \frac{F^{(n)}(y)}{n!} x^n = \frac{e^y}{n!} x^n$ for $y \in (0,x)$ (taking $x>0$).

Let's find $\sqrt{e} = e^{\frac{1}{2}}$ with error less than 0.0001 $\left(\frac{1}{10000}\right)$.

$R_n\left(\frac{1}{2}\right) = \frac{e^y}{n!\,2^n}$, $y \in \left(0, \frac{1}{2}\right)$. With a crude estimate, $e<3$, so $e^y < e^{\frac{1}{2}} < 3^{1/2} < 2$.

Thus $\left|R_n\left(\frac{1}{2}\right)\right| < \frac{1}{n!\,2^{n-1}}$. $\frac{1}{n!\,2^n} < \frac{1}{10000} \iff n!\,2^{n-1} > 10000$. Take $n=6$. Then error is less than $\frac{1}{23040}$.

$\Rightarrow \sqrt{e} \approx \sum_{k=0}^{5} \frac{1}{k!}\left(\frac{1}{2}\right)^k = 1 + \frac{1}{2} + \frac{1}{8} + \frac{1}{48} + \frac{1}{384} + \frac{1}{3840}$.