MATH 2052 Lecture 21 24 March 2026

Note: Near the end of the course, we will see an example of a function f whose Taylor series does not converge to f on any open interval.

Ex. $\lim_{x\to 0} \frac{(e^x - 1 - x)(\cos x - 1)^2}{(\sin x - x)^2}$

We could use L'Hopital's rule 4 times, or

$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$

$\Rightarrow e^x - 1 - x = \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots$

$= x^2\left(\frac{1}{2!} + \frac{1}{3!}x + \dots\right)$

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

$\Rightarrow \cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

$= x^2\left(-\frac{1}{2!} + \frac{x^2}{4!} - \dots\right)$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$\Rightarrow \sin x - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

$= x^3\left(-\frac{1}{3!} + \frac{x^2}{5!} - \dots\right)$

So the limit becomes:

$\lim_{x\to 0} \frac{x^2\left(\frac{1}{2!} + \frac{x}{3!} + \dots\right)\left[x^2\left(-\frac{1}{2!} + \frac{x^2}{4!} - \dots\right)\right]^2}{\left[x^3\left(-\frac{1}{3!} + \frac{x^2}{5!} - \dots\right)\right]^2}$

$= \lim_{x\to 0} \frac{\cancel{x^6}\left(\frac{1}{2!} + \frac{x}{3!} + \dots\right)\left(-\frac{1}{2!} + \frac{x^2}{4!} - \dots\right)^2}{\cancel{x^6}\left(-\frac{1}{3!} + \frac{x^2}{5!} - \dots\right)^2}$

$= \frac{\left(\frac{1}{2!}\right)\left(-\frac{1}{2!}\right)^2}{\left(-\frac{1}{3!}\right)^2}$

$= \frac{9}{2}$

Fourier Series

Given a periodic function f, can we represent f as an infinite of the "simple" periodic functions $\sin x$ and $\cos x$?

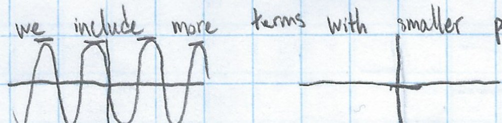
Def.

f is periodic with period $T = 2L > 0$ if $f(x+T) = f(x)$ for all x.

Note: $\cos\left(\frac{2\pi x}{T}\right) = \cos\left(\frac{\pi x}{L}\right)$ and $\sin\left(\frac{2\pi x}{T}\right) = \sin\left(\frac{\pi x}{L}\right)$ have period T.

Using scalar multiples of these two functions will give a very rough approximation of f.

If we include more terms with smaller periods, the sum may better approximate f.



We use terms of the form $a_n \cos\left(\frac{n\pi x}{L}\right)$ and $b_n \sin\left(\frac{n\pi x}{L}\right)$.

Suppose we are able to get $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$, $(\star)$

perhaps under idealized conditions. What can we say about a_n and b_n?

The following calculation is done assuming convergence and other idealized conditions to see what a_n and b_n must be, were such a series to actually converge.

Let $m \geq 0$ and multiply $(\star)$ by $\cos\left(\frac{m\pi x}{L}\right)$ and integrate from $-L$ to L:

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \int_{-L}^{L}\frac{a_0}{2}\cos\left(\frac{m\pi x}{L}\right)dx + \int_{-L}^{L}\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)\right]dx$$

$$= \frac{a_0}{2}\int_{-L}^{L}\cos\left(\frac{m\pi x}{L}\right)dx + \sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx + b_n\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx\right] \quad (\star\star)$$

When $m=0$, $(\star\star)$ is

$$\int_{-L}^{L} f(x)\,dx = \frac{a_0}{2}\int_{-L}^{L} 1\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)dx + b_n\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{a_0}{2}\,x\Big|_{-L}^{L} + \sum_{n=1}^{\infty}\left[\frac{a_n L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} - \frac{b_n L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}\right]$$

$$= \frac{a_0}{2}(L-(-L)) + \sum_{n=1}^{\infty}\left[\frac{a_n L}{n\pi}\underbrace{(\sin(n\pi)-\sin(-n\pi))}_{0} - \frac{b_n L}{n\pi}\underbrace{(\cos(n\pi)-\cos(-n\pi))}_{0}\right]$$

$$= a_0 L.$$

Thus $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$.

Note that for $k \geq 1$,

$$\int_{-L}^{L}\cos\left(\frac{k\pi x}{L}\right)dx = \frac{L}{k\pi}\sin\left(\frac{k\pi x}{L}\right)\Big|_{-L}^{L} = \frac{L}{k\pi}\left(\sin(k\pi)-\sin(k\pi)\right) = 0.$$

In particular, when $m \geq 1$ the first term of $(\star\star)$ is 0.

We add

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

to get $\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$

Letting $\alpha = \frac{n\pi x}{L}$ and $\beta = \frac{m\pi x}{L}$, we get

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\int_{-L}^{L}\cos\left(\frac{(n+m)\pi x}{L}\right) + \cos\left(\frac{(n-m)\pi x}{L}\right)\Big|_{-L}^{L}$$
$$= \begin{cases} 0 & \text{if } n \neq m \\ L & \text{if } n = m \end{cases}$$

Similarily, we add

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

to get $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$

Taking α, β as before, we get

$$\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\int_{-L}^{L}\left[\sin\left(\frac{(n+m)\pi x}{L}\right) + \sin\left(\frac{(n-m)\pi x}{L}\right)\right]dx$$
$$= 0 \quad \text{(by symmetry of sin)}$$

Using these in $(\star\star)$, we get $\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx = a_m L$

Thus $a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$

One can do this entire calculation again to obtain $b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx$ (multiplying by $\sin\left(\frac{m\pi x}{L}\right)$).

If such a series has any chance of converging, we must use these a_m and b_m for its coefficients.

Def.

Let f be periodic with period $T = 2L$.

The Fourier series for f is

$$\frac{a_0}{2} = \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$$

with $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$, and

$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$.

Exercise:

$$\int_{b}^{b+2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad \text{and} \quad \int_{b}^{b+2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

so we can integrate over any interval of width $T = 2L$ to get a_n, b_n.

Ex. Define $f(x)$ by $f(x) = x^2$ for $-1 \le x < 1$, $f(x+2) = f(x)$ for all x.

$T = 2$, so $L = \frac{T}{2} = 1$.

$a_0 = 1 \int_{-1}^{1} x^2 \, dx = \frac{1}{3} x^3 \Big|_{-1}^{1} = \frac{2}{3}$

For $n \ge 1$,

$$
\begin{aligned}
a_n &= 1 \int_{-1}^{1} x^2 \cos(n\pi x)\, dx \\
&= \frac{x^2}{n\pi} \sin(n\pi x) \Big|_{-1}^{1} - \frac{2}{n\pi} \int_{-1}^{1} x \sin(n\pi x)\, dx \qquad \text{(integration by parts)} \\
&= 0 - \frac{2}{n\pi} \left[-\frac{x}{n\pi} \cos(n\pi x) \Big|_{-1}^{1} + \frac{1}{n\pi} \int_{-1}^{1} \cos(n\pi x)\, dx \right] \\
&= \frac{2}{n^2\pi^2} \left[\cos(n\pi) - (-\cos(-n\pi)) \right] + \frac{1}{n^2\pi^2} \sin(n\pi x) \Big|_{-1}^{1} \\
&= \frac{2}{n^2\pi^2} 2\cos(n\pi) + 0 \\
&= \frac{4(-1)^n}{n^2\pi^2} \qquad \left(\cos(n\pi) = \begin{cases} 1 & \text{if } n \text{ even} \\ -1 & \text{if } n \text{ odd} \end{cases} \right)
\end{aligned}
$$

Similarly,

$$b_n = 1 \int_{-1}^{1} x^2 \sin(n\pi x)\, dx = 0.$$

The Fourier series for f is

$$\frac{1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2\pi^2} \cos(n\pi x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x).$$