Note:

f is even $\Leftrightarrow f(-x) = f(x) \ \forall x$

f is odd $\Leftrightarrow f(-x) = -f(x) \ \forall x$

If f is odd, then

$$\begin{aligned}\int_{-L}^{L} f(x)\,dx &= \int_{-L}^{0} f(x)\,dx + \int_0^L f(x)\,dx \\ &= -\int_L^0 f(-y)\,dy + \int_0^L f(x)\,dx \\ &= -\int_L^0 -f(y)\,dx + \int_0^L f(x)\,dx \\ &= \int_L^0 f(x)\,dx + \int_0^L f(x)\,dx \\ &= -\int_0^L f(x)\,dx + \int_0^L f(x)\,dx \\ &= 0.\end{aligned}$$

Similarily, if f is even, then $\int_{-L}^{L} f(x)\,dx = 2\int_0^L f(x)\,dx$.

Exercise:

If f and g are even, then fg is even.

If f and g are odd, then fg is even.

If f is even and g is odd, then fg is odd.

Ex. Suppose f is even.

Then $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = 0$

Similarily, if f is odd, then

$a_n = 0$, and

$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Def.

If a periodic function f is even, then its Fourier series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)$ with $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ is called a <u>Fourier cosine series</u>.

If f is odd, then its Fourier series $\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$ with $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ is called a <u>Fourier sine series</u>.

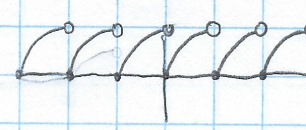
Ex. $f(x) = x^2$ for $-1 \le x < 1$, $f(x+2) = f(x) \ \forall x$. Since f is even, $b_n = 0$.

Recall:

A function is piecewise continuous on $[a,b]$ if there is a partition P on $[a,b]$ such that f is uniformly continuous on each subinterval (t_{k-1}, t_k).

We say that f is piecewise continuous if it is piecewise continuous on every interval $[a,b]$.

Ex. $f(x) = \sqrt{x}$ for $0 \le x < 1$ and $f(x+1) = f(x)$ $\forall x$.



f is piecewise continuous.

$f'(x) = \frac{1}{2\sqrt{x}}$ for $0 < x < 1$ and $f'(x+1) = f'(x)$ $\forall x$.

f' is not piecewise continuous since as x approaches any integer from above, $f'(x) \to \infty$.

Note For convenience, we write $f(x+) = \lim_{t \to x^+} f(t)$ and $f(x-) = \lim_{t \to x^-} f(t)$.

Thm Fourier Convergence Theorem

Let f be periodic with period $T = 2L$.

If f and f' are piecewise continuous, then the Fourier series for f converges (pointwise) to $\frac{1}{2}(f(x+) + f(x-))$.

In particular, if f is continuous at x, then $f(x+) = f(x) = f(x-)$, and so the Fourier series for f converges to f at x.

If f has a jump discontinuity at x, then the Fourier series converges to the point halfway between the left and right limits at x.

Proof: omitted for this lecture, will be proved later.

Ex. $f(x) = x^2$ for $-1 \le x < 1$ and $f(x+2) = f(x)$ $\forall x$.

f is piecewise continuous.

$f'(x) = 2x$ for $-1 < x < 1$ and $f'(x+2) = f'(x)$ $\forall x$.

f' is also piecewise continuous, so the Fourier convergence theorem can be used.

We found the Fourier series for f. Since f is continuous, by the Fourier convergence theorem it converges to f.

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x).$$

As a neat consequence, let $x = 0$.

Since $f(0) = 0^2 = 0$, we get

$$\frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = 0$$

$$\Rightarrow -\frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{1}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

By absolute convergence, we have

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{2}{4} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n^2} = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = 2\left(\frac{\pi^2}{12}\right) = \frac{\pi^2}{6}$. (The Basel problem.)

Ex. Let $f(x) = x$ if $0 \leq x < 1$, and $f(x+1) = f(x)$ $\forall x$.

f is piecewise continuous.

$f'(x) = 1$ for $0 < x < 1$, $f'(x+1) = f'(x)$ is also piecewise continuous.

By the Fourier convergence theorem, the Fourier series for f converges to $f(x)$ when x is not an integer.

If x is an integer, the series converges to $\frac{1}{2}$.