MATH 2052 — Lecture 23 — 31 March 2026

Note:

We have been forming Fourier series for periodic functions.
If f is a function defined on an interval, then we can extend it to a periodic function to obtain a Fourier series for f.

T-periodic extension

For f defined on $[a,b]$, set $T = b-a$ and define $\tilde{f}(x) = f(x)$ for $a \le x < b$ and $\tilde{f}(x+T) = \tilde{f}(x)$ $\forall x$.
Then the Fourier series for f is the Fourier series for $\tilde{f}$ with coefficients

$$a_n = \frac{1}{L}\int_a^b \tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)dx$$
$$= \frac{1}{L}\int_a^b f(x)\cos\left(\frac{n\pi x}{L}\right)dx, \text{ and}$$
$$b_n = \frac{1}{L}\int_a^b \tilde{f}(x)\sin\left(\frac{n\pi x}{L}\right)dx$$
$$= \frac{1}{L}\int_a^b f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

If f is defined on an interval $[0,L]$, then we could do as above, but we have more options.
Alternatively, we can first extend f to f_1 defined on $[-L,L]$ with f_1 even.
We can then form the $2L$-periodic extension $\tilde{f}$ of f_1.
Since $\tilde{f}$ is also even, the Fourier series for $\tilde{f}$ will have all $b_n = 0$.
The Fourier cosine series for f is the Fourier (cosine) series $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)$ with coefficients

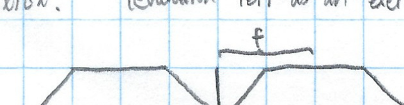
$$a_n = \frac{2}{L}\int_0^L \tilde{f}(x)\cos\left(\frac{n\pi x}{L}\right)dx$$
$$= \frac{2}{L}\int_0^L f_1(x)\cos\left(\frac{n\pi x}{L}\right)dx$$
$$= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \quad \text{(since on } (0,L),\ \tilde{f}(x) = f_1(x) = f(x).)$$

Similarly, we can instead extend f on $[0,L]$ to f_1 on $[-L,L]$ with f_1 odd. (note: we will redefine $f_1(0)=0$, which does not change what the Fourier series converges to, nor its coefficients).
Then let $\tilde{f}$ be the $2L$-periodic extension of f_1, so $\tilde{f}$ is odd and hence all $a_n = 0$.
The Fourier sine series for f is the Fourier (sine) series $\sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$ with coefficients

$$b_n = \frac{2}{L}\int_0^L \tilde{f}(x)\sin\left(\frac{n\pi x}{L}\right)dx$$
$$= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

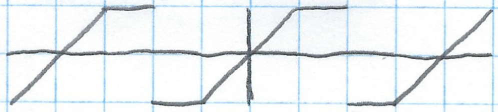
Ex. Let $f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 \le x < 2. \end{cases}$

2-periodic extension: $T = 2-0 = 2$, $L = \frac{T}{2} = 1$.
By Fourier convergence theorem, the Fourier series converges to f on $(0,2)$. At $x = 0, 2, \ldots$, it converges to $\frac{1}{2}$.
Fourier series is $\frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)\right]$.
For $a_n = \int_0^2 f(x)\cos(n\pi x)\,dx = \int_0^1 x\cos(n\pi x)\,dx + \int_1^2 \cos(n\pi x)\,dx$. (evaluation left as an exercise)
$b_n = \int_0^2 f(x)\sin(n\pi x)\,dx$.



Even extension (cosine series): $L = 2$, $[0,2] = [0,L]$.
Since $\tilde{f}$ is continuous, the Fourier series for f will converge to f on $[0,2]$.
Fourier cosine series is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right)$
for $a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{L}\right)dx$. (evaluation also left as an exercise)

Odd extension (sine series) $L=2$



Fourier sine series converges to f on $[0,2)$. $x=2 \to 0$.

Fourier sine series is $\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$

for $b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$

$= \int_0^1 x\sin\left(\frac{n\pi x}{2}\right)dx + \int_1^0 \sin\left(\frac{n\pi x}{2}\right)dx$

$\int u\,dv = uv - \int v\,du$, $u = x$, $dv = \sin\left(\frac{n\pi x}{2}\right)$

$du = 1$, $v = -\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)$.

$= -\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx - \frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Big|_1^2$

$= -\frac{2}{n\pi}\cancel{\cos\left(\frac{n\pi}{2}\right)} - 0 + \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)\Big|_0^1 - \frac{2}{n\pi}\left[\cos(n\pi) - \cancel{\cos\left(\frac{n\pi}{2}\right)}\right]$

$= \frac{4}{n^2\pi^2}\left[\sin\left(\frac{n\pi}{2}\right) - \sin 0\right] - \frac{2}{n\pi}(-1)^n$

$= \frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2(-1)^n}{n\pi}$.