Note: we will eventually prove that $L(f) \leq U(f)$. Note that this does not immediately follow from ⊛ defined in the last class. For example, we have $L(f, P_1) \leq U(f, P_2)$, $L(f, P_2) \leq U(f, P_2)$, but what if $L(f, P_1) > U(f, P_2)$? We'd then have $L(f) = \sup\{L(f,P)\} > \inf\{U(f,P)\} = U(f)$. We will show this can't happen, within the next lecture, but it requires some proof.

Def.

If $L(f) = U(f)$, then we say f is integrable on $[a,b]$, and write

$$L(f) = U(f) = \int_a^b f(x)\,dx \doteq \int_a^b f.$$

If $L(f) \neq U(f)$, then we say f is not integrable on $[a,b]$.

Ex. Consider $f(x) = c$, for $c \in \mathbb{R}$, on $[a,b]$.

For any subinterval $[t_{k-1}, t_k]$ of any partition P,

$$M(f, [t_{k-1}, t_k]) = c = m(f, [t_{k-1}, t_k]).$$

Thus for any partition P,

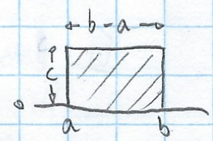
$$\begin{aligned} U(f,P) &= \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])(t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} c(t_k - t_{k-1}) \\ &= c(\cancel{t_1} - t_0 + \cancel{t_2} - \cancel{t_1} + \dots + t_n - \cancel{t_{n-1}}) \\ &= c(t_n - t_0) \\ &= c(b-a) \end{aligned}$$

$$\begin{aligned} L(f,P) &= \sum_{k=1}^{n} m(f, [t_{k-1}, t_k])(t_k - t_{k-1}) \\ &= c(b-a) \end{aligned}$$

Thus $U(f) = \inf\{c(b-a) : P \text{ is a partition of } [a,b]\}$ (independent of P)
$= c(b-a)$,

and $L(f) = \sup\{c(b-a) : P \text{ is a partition of } [a,b]\}$ (independent of P)
$= c(b-a)$.

Since $U(f) = L(f)$, $f(x) = c$ is integrable on $[a,b]$ with $\int_a^b c\,dx = c(b-a)$.
This is indeed the area under the "curve"/line $y = c$, bounded by $x = a$, $x = b$.



Note: If $s = \sum_{k=1}^{n} k = 1 + 2 + \dots + n$, then

$$\begin{aligned} s &= n + (n-1) + \dots + (n-(n-1)) \\ 2s &= (n+1) + (n+1) + \dots + (n+1) \\ &= n(n+1) \\ \Rightarrow s &= \frac{n(n+1)}{2}. \end{aligned}$$

Ex. Consider $f(x) = x$ on $[0,b]$ with $b=0$. For any subinterval $[t_{k-1}, t_k]$ of any partition P,

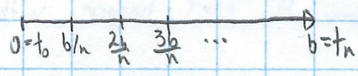
$M(f, [t_{k-1}, t_k]) = t_k$, and

$m(f, [t_{k-1}, t_k]) = t_{k-1}$.

Thus $U(f,P) = \sum_{k=1}^{n} t_k(t_k - t_{k-1})$, and

$L(f,P) = \sum_{k=1}^{n} t_{k-1}(t_k - t_{k-1})$.

Consider the family of partitions with $t_k = \frac{kb}{n}$.



For these P,

$$U(f,P) = \sum_{k=1}^{n} \frac{kb}{n}\left(\frac{kb}{n} - \frac{(k-1)b}{n}\right)$$
$$= \frac{b^2}{n^2}\sum_{k=1}^{n} k = \frac{b^2}{n^2}\cdot\frac{n(n+1)}{2} = \frac{b^2}{2}\left(\frac{n^2+n}{n^2}\right)$$

Since $\frac{n^2+n}{n^2} > \frac{n^2}{n^2} = 1$ and $\lim_{n\to\infty} \frac{n^2+n}{n^2} = 1$,

we have $\inf\left\{\frac{b^2}{2}\left(\frac{n^2+n}{n^2}\right) : n \in \mathbb{N}\right\} = \frac{b^2}{2}$.

Hence $U(f) \le \frac{b^2}{2}$ (inf. of $U(f,P)$ over all partitions).

Similarly, for the above partitions P,

$$L(f,P) = \sum_{k=1}^{n} \frac{(k-1)b}{n}\left(\frac{kb}{n} - \frac{(k-1)b}{n}\right)$$
$$= \frac{b^2}{n^2}\sum_{k=1}^{n}(k-1)$$
$$= \frac{b^2}{n^2}\left(\frac{n(n+1)}{2} - n\right)$$
$$= \frac{b^2}{2}\left(\frac{n^2-n}{n^2}\right).$$

Since $\frac{n^2-n}{n^2} < \frac{n^2}{n^2} = 1$ and $\lim_{n\to\infty}\frac{n^2-n}{n^2} = 1$,

we have $\sup\left\{\frac{b^2}{2}\left(\frac{n^2-n}{2}\right) : n \in \mathbb{N}\right\} = \frac{b^2}{2}$.

Thus $L(f) \ge \frac{b^2}{2}$ (sup. of $L(f,P)$ over all partitions P.

Thus we have $\frac{b^2}{2} \le L(f) \le U(f) \le \frac{b^2}{2}$, ($L(f) \le U(f)$ to be proved)

which implies $L(f) = U(f) = \frac{b^2}{2}$.

Thus $f(x) = x$ is integrable on $[0,b]$ with $\int_0^b x\,dx = \frac{b^2}{2}$.

Note: $f(x) = x^2$ on $[0,b]$ is shown in our textbook.

Ex. Consider $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ (1 if rational, 0 if irrational) on $[a,b]$ with $b>a$.

For any subinterval $[t_{k-1}, t_k]$ of any partition P,

$M(f, [t_{k-1}, t_k]) = 1$, and

$m(f, [t_{k-1}, t_k]) = 0$. (because between any 2 real numbers, there exists another rational and irrational number between them).

Thus for any P,

$U(f,P) = \sum_{k=1}^{n}(1)(t_k - t_{k-1}) = b-a$, and

$L(f,P) = \sum_{k=1}^{n}(0)(t_k - t_{k-1}) = 0$.

Hence $U(f) = \inf\{b-a : P \text{ is a partition of } [a,b]\} = a-b$,

$L(f) = \sup\{0 : P \text{ is a partition of } [a,b]\} = 0$.

Since $U(f) \ne L(f)$, f is not integrable on $[a,b]$.

Lemma: 32.2

Let f be a bounded function on $[a,b]$. If P and Q are partitions of $[a,b]$ with $P \subseteq Q$ (i.e. Q is a "finer" partition), then

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

Proof:

We've already noted that $L(f,Q) \leq U(f,Q)$. We will show $L(f,P) \leq L(f,Q)$, as the proof for the remaining inequality is similar.

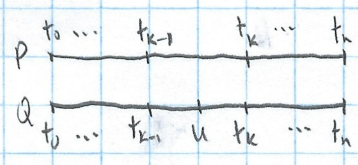
Let us assume that Q has only one more point than P (for then we could use induction, or apply this repeatedly, to get the general result).

If P consists of

$$a = t_0, t_1, t_2, \ldots, t_n = b,$$

let Q consist of

$$a = t_0, t_1, \ldots, t_{k-1}, u, t_k, \ldots, t_n = b$$



for some integer k with $1 \leq k \leq n$.

Then most terms in $L(f,P)$ and $L(f,Q)$ are the same.

In particular, we have

$$L(f,Q) - L(f,P) = m(f,[t_{k-1},u])(u-t_{k-1}) + m(f,[u,t_k])(t_k-u) - m(f,[t_k,t_{k-1}])(t_k-t_{k-1}). \quad ⊛$$

Since $[t_{k-1},u] \subseteq [t_{k-1},t_k]$, we have

$$\inf\{f(x) : x \in [t_{k-1},u]\} \geq \inf\{f(x) : x \in [t_{k-1},t_k]\}, \text{ and so}$$

$$m(f,[t_{k-1},u]) \geq m(f,[t_{k-1},t_k]).$$

Similarly, $m(f,[u,t_k]) \geq m(f,[t_{k-1},t_k])$.

Thus by ⊛ we have

$$\begin{aligned} L(f,Q) - L(f,P) &\geq m(f,[t_{k-1},t_k])(u-t_{k-1}) + m(f,[t_{k-1},t_k])(t_k-u) - m(f,[t_{k-1},t_k])(t_k-t_{k-1}) \\ &= m(f,[t_{k-1},t_k])(u-t_{k-1}+t_k-u-(t_k-t_{k-1})) \\ &= 0. \end{aligned}$$

Hence $L(f,P) \leq L(f,Q)$. □