<u>Lemma</u>: 32.3

Let f be bounded on $[a,b]$, and let P and Q be partitions of $[a,b]$. Then $L(f,P) \le U(f,Q)$.

<u>Proof</u>:

$P \cup Q$ (union of P and Q) is a partition of $[a,b]$ with $P \subseteq P \cup Q$ and $Q \subseteq P \cup Q$. Thus lemma 32.2 implies

$$L(f,P) \le L(f, P\cup Q) \le U(f, P \cup Q) \le U(f,Q). \quad \square$$

<u>Thm.</u> 32.4

Let f be bounded on $[a,b]$. Then $L(f) \le U(f)$.

<u>Proof</u>:

Fix an arbitrary partition P of $[a,b]$. By lemma 32.3, $L(f,P)$ is a lower bound of $\{U(f,Q) : Q$ is a partition of $[a,b]\}$.

Thus for any P, $L(f,P) \le \underline{U(f)}$ (inf of $\{U(f,Q)\}$).

This implies that $U(f)$ is an upper bound of $\{L(f,P) : P$ is a partition of $[a,b]\}$, and so $\underline{L(f)} \le U(f)$. (sup of $\{L(f,P)\}$). $\square$

<u>Thm.</u> 32.5

A bounded function f on $[a,b]$ is integrable if and only if

$\forall \varepsilon > 0$, there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$.

<u>Proof</u>:

Suppose f is integrable on $[a,b]$ and let $\varepsilon > 0$ be given. Then there exists partitions P_1 and P_2 such that

$$L(f,P_1) > L(f) - \frac{\varepsilon}{2} \quad \text{and} \quad U(f,P_2) < U(f) + \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$ and use lemma 32.2 to get

$$\begin{aligned} U(f,P) - L(f,P) &\le U(f,P_2) - L(f,P_1) \\ &< U(f) + \frac{\varepsilon}{2} - L(f) + \frac{\varepsilon}{2} && \text{(by ⊛)} \\ &= U(f) - L(f) + \varepsilon \\ &= \varepsilon && \text{(since } f \text{ is integrable, therefore } U(f) = L(f).) \end{aligned}$$

Now we prove the reverse direction. Suppose for $\varepsilon > 0$, $U(f,P) - L(f,P) < \varepsilon$ holds for some partition P.

Then
$$\begin{aligned} U(f) &\le U(f,P) \\ &= U(f,P) - L(f,P) + L(f,P) && \text{(add and subtract same term.)} \\ &< \varepsilon + L(f,P) \\ &\le \varepsilon + L(f). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $U(f) \le L(f)$ (for if $U(f) > L(f)$, then letting $\varepsilon = U(f) - L(f)$, we have $U(f) < \varepsilon + L(f) = U(f)$, a contradiction).

Since by theorem 32.4 we have $L(f) \le U(f)$, it follows that $L(f) = U(f)$. Hence f is integrable. $\square$

Def.

A function is monotone on an interval if it is either increasing on the interval, or decreasing on the interval.

Thm. 33.1

Every monotone function f on $[a,b]$ is integrable.

Proof:

We prove the increasing case (the decreasing case is an exercise). We may assume $f(a) < f(b)$, for otherwise f is constant and hence integrable.

Since $f(a) \le f(x) \le f(b)$ for all $x \in [a,b]$, f is bounded on $[a,b]$. ← This is called the mesh of P.

Let $\varepsilon > 0$ be given. Take P a partition of $[a,b]$ with $\underline{\max\{t_k - t_{k-1} : 1 \le k \le n\}} < \frac{\varepsilon}{f(b)-f(a)}$.

Then
$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n} \left(M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\right)(t_k - t_{k-1}) \\ &= \sum_{k=1}^{n} (f(t_k) - f(t_{k-1}))(t_k - t_{k-1}) \\ &< \sum_{k=1}^{n} (f(t_k) - f(t_{k-1})) \frac{\varepsilon}{f(b)-f(a)} \\ &= \frac{\varepsilon}{f(b)-f(a)} \left(f(t_1) - f(t_0) + f(t_2) - f(t_1) + \dots + f(t_n) - f(t_{n-1})\right) \\ &= \frac{\varepsilon}{f(b)-f(a)} (f(t_n) - f(t_0)) \quad \text{(telescoping)} \\ &= \frac{\varepsilon}{f(b)-f(a)} (f(b) - f(a)) \\ &= \varepsilon. \end{aligned}$$

Thus by thm. 32.5, f is integrable on $[a,b]$. □

Ex. The following functions are integrable on a closed interval in their domain:

$\sqrt{x}$, $\frac{1}{x}$, $\frac{1}{x^n}$, $\ln x$, $\frac{1}{\ln x}$, e^x, $\lfloor x \rfloor$ (floor function), $\tan x$, so on.

Thm. 33.2

Every continuous function f on $[a,b]$ is integrable.

Proof:

Let $\varepsilon > 0$ be given. Since f is continuous on the closed interval $[a,b]$, by thm. 19.2 (MATH 1052), it is uniformly continuous on $[a,b]$.

Thus there is a $\delta > 0$ such that $x, y \in [a,b]$ and $|x-y| < \delta$ imply $|f(x) - f(y)| < \frac{\varepsilon}{b-a}$. (*)

Let P be a partition with $\max\{t_k - t_{k-1} : 1 \le k \le n\} < \delta$.

By thm. 18.1 (MATH 1052), f assumes its maximum and minimum on each closed interval $[t_{k-1}, t_k]$.

Thus (*) implies

$\underline{M(f,[t_{k-1},t_k])} - \underline{m(f,[t_{k-1},t_k])} < \frac{\varepsilon}{b-a}$. $\quad M(\dots) = f(x)$ and $m(\dots) = f(y)$ for some $x, y \in [t_{k-1}, t_k]$.

Thus
$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{k=1}^{n} \left(M(f,[t_{k-1},t_k]) - m(f,[t_{k-1},t_k])\right)(t_k - t_{k-1}) \\ &< \sum_{k=1}^{n} \frac{\varepsilon}{b-a}(t_k - t_{k-1}) \\ &= \frac{\varepsilon}{b-a} \sum_{k=1}^{n} (t_k - t_{k-1}) \\ &= \frac{\varepsilon}{b-a}(b-a) \quad \text{(telescoping series)} \\ &= \varepsilon. \end{aligned}$$

Thus theorem 32.5 implies f is integrable on $[a,b]$. □

Ex. The following functions are continuous on a closed interval in it's domain:
any polynomials or rational functions, $\sin x$, $\cos x$, $|x|$, e^{-x^2}, etc.