Thm: 33.3

Let f and g be integrable on $[a,b]$ and let c be a constant. Then

(i) cf is integrable with $\int_a^b cf = c\int_a^b f$.

(ii) $f+g$ is integrable with $\int_a^b (f+g) = \int_a^b f + \int_a^b g$.

Note one can show that fg is integrable but there is no nice formula in terms of $\int_a^b f$ and $\int_a^b g$.

To prove this, we note the following useful results:

(a) if $c>0$, then $\inf\{cs : s\in S\} = c\inf S$ and $\sup\{cs : s\in S\} = c\sup S$.

(b) $\inf\{-s : s\in S\} = -\sup S$ and $\sup\{-s : s\in S\} = -\inf S$.

(c) $\inf\{f(x)+g(x) : x\in S\} \geq \inf\{f(x) : x\in S\} + \inf\{g(x) : x\in S\}$, and
$\sup\{f(x)+g(x) : x\in S\} \leq \sup\{f(x) : x\in S\} + \sup\{g(x) : x\in S\}$.

Proof:

(i) If $c=0$, the result is clear. First suppose $c>0$. Then for any subinterval we have

$$M(cf, [t_{k-1}, t_k]) = cM(f, [t_{k-1}, t_k]), \text{ and}$$
$$m(cf, [t_{k-1}, t_k]) = cm(f, [t_{k-1}, t_k]) \quad \text{(by result (a))}.$$

Thus for any partition P, we have

$$U(cf, P) = cU(f,P), \text{ and}$$
$$L(cf, P) = cL(f,P).$$

Result (a) then implies $U(cf) = cU(f)$ and $L(cf) = cL(f)$. Then

$$\begin{aligned} L(cf) &= cL(f) \\ &= cU(f) \quad \text{(since } f \text{ is integrable)} \\ &= U(cf), \end{aligned}$$

and hence cf is integrable, with $\int_a^b cf = U(cf) = cU(f) = c\int_a^b f$ (for $c>0$). ⊛

Now suppose $c=-1$. Then result (b) implies

$$M(-f, [t_{k-1}, t_k]) = -m(f, [t_{k-1}, t_k]).$$

Thus $U(-f,P) = -L(f,P)$. Thus

$$\begin{aligned} U(-f) &= \inf\{U(-f,P) : P \text{ is a partition of } [a,b]\} \\ &= \inf\{-L(f,P) : P \text{ is a partition of } [a,b]\} \\ &= -\sup\{L(f,P) : P \text{ is a partition of } [a,b]\} \\ &= -L(f). \end{aligned}$$

One may similarly show that $L(-f) = -U(f)$.

Then
$$\begin{aligned} U(-f) &= -L(f) \\ &= -U(f) \\ &= -(-L(f)) \\ &= L(-f), \end{aligned}$$

and hence $-f$ is integrable, with $\int_a^b -f = U(-f) = -U(f) = -\int_a^b f$. ⊛⊛

Finally, suppose $c<0$. Then

$$\begin{aligned} \int_a^b cf = \int_a^b -(-c)f &= -\int_a^b (-c)f \\ &= -(-c)\int_a^b f \\ &= c\int_a^b f. \end{aligned}$$

(ii) We use thm. 32.5. Let $\varepsilon > 0$ be given. Then there are partitions P_1 and P_2 of $[a,b]$ with

$U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$, and

$U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2}$.

Letting $P = P_1 \cup P_2$ and using lemma 32.2 yields

$U(f,P) - L(f,P) < \frac{\varepsilon}{2}$, and

$U(g,P) - L(g,P) < \frac{\varepsilon}{2}$.

By result (c), we have

$$m(f+g,[t_{k-1},t_k]) \geq m(f,[t_{k-1},t_k]) + m(g,[t_{k-1},t_k]),$$

and so $L(f+g,P) \geq L(f,P) + L(g,P)$.

Similarily, $U(f+g,P) \leq U(f,P) + U(g,P)$.

$$\begin{aligned} \text{Thus } U(f+g,P) - L(f+g,P) &\leq U(f,P) + U(g,P) - L(f,P) - L(g,P) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

$$\begin{aligned} \text{Thus by theorem 32.5, } f+g \text{ is integrable, with } \int_a^b (f+g) &= U(f+g) \\ &\leq U(f+g,P) \\ &\leq U(f,P) + U(g,P) \\ &< L(f,P) + L(g,P) + \varepsilon \\ &\leq L(f) + L(g) + \varepsilon \\ &= \int_a^b f + \int_a^b g + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\int_a^b (f+g) \leq \int_a^b f + \int_a^b g$.

$$\begin{aligned} \text{Also, } \int_a^b (f+g) &= L(f+g) \\ &\geq L(f+g,P) \\ &\geq L(f,P) + L(g,P) \\ &> U(f,P) + U(g,P) - \varepsilon \\ &\geq U(f) + U(g) - \varepsilon \\ &= \int_a^b f + \int_a^b g - \varepsilon, \end{aligned}$$

which implies $\int_a^b (f+g) \geq \int_a^b f + \int_a^b g$.

Thus $\int_a^b f + \int_a^b g$. □

<u>Thm.</u> 33.4

(i) If f and g are integrable on $[a,b]$ and $f(x) \leq g(x)$ $\forall x \in [a,b]$, then $\int_a^b f \leq \int_a^b g$.

(ii) If g is a <u>continuous</u> nonnegative function on $[a,b]$ with $\int_a^b g = 0$, then $g(x) = 0$ $\forall x \in [a,b]$.

<u>Proof:</u>

(i) Thm 33.3 states that $h = g - f$ is integrable on $[a,b]$.

Since $h(x) \geq 0$ $\forall x \in [a,b]$, we have $L(h,P) \geq 0$ for any partition P of $[a,b]$. Thus

$\int_a^b h = L(h) \geq 0$.

Thus since $g = f + h$, we have

$$\begin{aligned} \int_a^b g &= \int_a^b (f+h) \\ &= \int_a^b f + \int_a^b h && \text{(by thm 33.3)} \\ &\geq \int_a^b f. && \text{(since } \textstyle\int_a^b h \geq 0) \end{aligned}$$

Thm 33.5

If f is integrable on $[a,b]$, then $|f|$ is integrable with $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Proof:

Proof that $|f|$ is integrable will be done in tutorial.

Since $-|f| \leq f \leq |f|$, theorems 33.4 and 33.3 imply

$$-\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f|,$$

which implies $\left|\int_a^b f\right| \leq \int_a^b |f|$. □

Thm. 33.6

Let f be defined on $[a,b]$ and let $a<c<b$. If f is integrable on $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$ with $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof:

Next class.