Thm. 33.6

Let f be defined on $[a,b]$ and let $a<c<b$. If a is integrable on $[a,c]$ and $[c,b]$, then f is integrable on $[a,b]$ with $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof:

f is bounded on $[a,c]$ and on $[c,b]$, so it is bounded on $[a,b]$.

Let $\varepsilon > 0$ be given. By theorem 32.5, there are partitions P_1 and P_2 of $[a,c]$ and $[c,b]$ respectively with $U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$, and $U(f,P_2) - L(f,P_2) < \frac{\varepsilon}{2}$.

Then $P = P_1 \cup P_2$ is a partition of $[a,b]$ with $U(f,P) = U(f,P_1) + U(f,P_2)$ and $L(f,P) = L(f,P_1) + L(f,P_2)$.

Thus $U(f,P) - L(f,P) = U(f,P_1) + U(f,P_2) - L(f,P_1) - L(f,P_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, and hence f is integrable on $[a,b]$.

Also,
$$\begin{aligned}\int_a^b f \le U(f,P) = U(f,P_1) + U(f,P_2) &< L(f,P_1) + L(f,P_2) + \varepsilon \\ &\le \int_a^c f + \int_c^b f + \varepsilon\end{aligned}$$

for any $\varepsilon > 0$, and hence $\int_a^b f \le \int_a^c f + \int_c^b f$. Similarly,
$$\begin{aligned}\int_a^b f \ge L(f,P) = L(f,P_1) + L(f,P_2) &> U(f,P_1) + U(f,P_2) - \varepsilon \\ &\ge \int_a^c f + \int_c^b f - \varepsilon\end{aligned}$$

for any $\varepsilon > 0$, and hence $\int_a^b f \ge \int_a^c f + \int_c^b f$.

Thus $\int_a^b f = \int_a^c f + \int_c^b f$. $\square$

Def.

A function f is <u>piecewise monotonic</u> on $[a,b]$ if there is a partition P of $[a,b]$ such that f is monotonic on each subinterval (t_{k-1}, t_k).

f is piecewise continuous on $[a,b]$ such that f is uniformly continuous on each open subinterval (t_{k-1}, t_k).

Remark

Let f be uniformly continuous on (a,b). Then $f(a+\frac{1}{n})$ and $f(b-\frac{1}{n})$ must converge, say to L_1 and L_2 respectively. If we extend f to f_1 by setting $f_1(x) = f(x)$ for $x \in (a,b)$, $f_1(a) = L_1$ and $f_1(b) = L_2$, then one can show that f_1 is a continuous extension of f to $[a,b]$.

Thm. 33.8

If f is piecewise continuous on $[a,b]$, or if f is a bounded piecewise monotonic function on $[a,b]$, then f is integrable on $[a,b]$.

Proof

Suppose f is piecewise continuous on $[a,b]$ and let P consist of the subintervals on which f is uniformly continuous. Consider a subinterval (t_{k-1}, t_k). By our remark above, we can extend f to a continuous function f_k on $[t_{k-1}, t_k]$. By thm. 33.2, f_k is integrable on $[t_{k-1}, t_k]$. We have $f = f_k$ except possibly on the endpoints. By exercise 32.7, f is integrable on $[t_{k-1}, t_k]$. Then thm. 33.6 implies f is integrable on $[a,b]$. (...)

Now suppose f is bounded and piecewise monotone on $[a,b]$, and let P be a partition for which f is monotone on each subinterval (t_{k-1}, t_k). We can then extend f to a monotone function f_k on $[t_{k-1}, t_k]$ (for example, if f is increasing on (t_{k-1}, t_k), let $f_k(t_k) = \sup\{f(x) : x \in (t_{k-1}, t_k)\}$, etc.)
Then by thm. 33.1, f_k is integrable on $[t_{k-1}, t_k]$, and so f is integrable on $[t_{k-1}, t_k]$, and so f is integrable there too (exercise 32.7). By thm. 33.6, f is integrable on $[a,b]$. $\square$

Thm 34.1: Fundamental Theorem of Calculus I

If g is continuous on $[a,b]$ and differentiable on (a,b), and g' is integrable on $[a,b]$, then $\int_a^b g' = g(b) - g(a)$.

Proof:

Let $\varepsilon > 0$ be given. By thm. 32.5, there is a partition P of $[a,b]$ with

$U(g', P) - L(g', P) < \varepsilon$. (*)

Applying the mean value theorem to g on each interval $[t_{k-1}, t_k]$ yields $x_k \in (t_{k-1}, t_k)$ such that

$g'(x_k) = \frac{g(t_k) - g(t_{k-1})}{t_k - t_{k-1}}$,

and so $g(t_k) - g(t_{k-1}) = g'(x_k)(t_k - t_{k-1})$. Summing over all k yields

$\sum_{k=1}^{n} (g(t_k) - g(t_{k-1})) = \sum_{k=1}^{n} g'(x_k)(t_k - t_{k-1}) = g(b) - g(a)$.

Thus since $m(g', [t_{k-1}, t_k]) \le g'(x_k) \le M(g', [t_{k-1}, t_k])$, we have $L(g', P) \le g(b) - g(a) \le U(g', P)$. (*)(*)

On the other hand, we have $L(g', P) \le \int_a^b g' \le U(g', P)$.

Subtracting (*)(*) from this and using (*) yields

$-\varepsilon < L(g', P) - U(g', P) \le \int_a^b g' - (g(b) - g(a)) \le U(g', P) - L(g', P) < \varepsilon$,

and so $0 \le |\int_a^b g' - (g(b) - g(a))| < \varepsilon$.

Since this holds for every $\varepsilon > 0$, we must have $|\int_a^b g' - (g(b) - g(a))| = 0$, and so

$\int_a^b g' - (g(b) - g(a)) = 0$

$\Rightarrow \int_a^b g' = g(b) - g(a)$. $\square$

Thus, if we are trying to integrate an integrable function f, we should look for a function F such that $F' = f$. Then $\int_a^b F$ is called an antiderivative of f.
It is common to write $F(x)\big|_a^b$ for $F(b) - F(a)$.

Ex. $\int_0^b x^3\,dx$. Can we find $F(x)$ with $F'(x) = x^3$?

$F(x) = \frac{1}{4}x^3$ works.

By FTC I, we have

$\int_0^b x^3\,dx = \frac{1}{4}x^3\,dx = \frac{1}{4}x^4\big|_0^b = \frac{1}{4}b^4 - 0 = \frac{1}{4}b^4$.

Note: Consider $\int_a^b f$. If it is unknown that f is integrable, but we do find F with $F' = f$, this does not imply that f is actually integrable.

For example, $F(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}) & \text{for } x \ne 0 \\ 0 & \text{for } x = 0. \end{cases}$

F is differentiable everywhere: $f = \begin{cases} 2x\sin(\frac{1}{x^2}) - 2\cos(\frac{1}{x^2})/x & \text{for } x \ne 0 \\ 0 & \text{for } x = 0 \end{cases}$

f is not bounded near $0 \to$ not integrable.

For a bounded counterexample, see Volterra's function (not course material).