Ex. In general, if $n \neq -1$, and for either $a,b>0$ or $a,b<0$,

$$\int_a^b x^n\,dx = \frac{1}{n+1}x^{n+1}\Big|_a^b = \frac{1}{n+1}b^{n+1} - \frac{1}{n+1}a^{n+1}.$$

For example, $\int_0^1 \sqrt{x}\,dx = \int_0^1 x^{\frac{1}{2}}\,dx$

$$= \frac{2}{3}x^{\frac{3}{2}}\Big|_0^1 = \frac{2}{3}.$$

For $n=-1$, for $a,b>0$,

$$\int_a^b x^{-1}\,dx = \int_a^b \frac{1}{x}\,dx = \ln x\Big|_a^b \qquad (\text{since } (\ln x)' = \tfrac{1}{x}.)$$
$$= \ln b - \ln a.$$

Note if $x<0$, $(\ln|x|)' = (\ln(-x))' = \frac{1}{-x}(-1) = \frac{1}{x}$,

and for $x>0$, $(\ln|x|)' = (\ln x)' = \frac{1}{x}$.

Thus if either $a,b>0$ or $a,b<0$, then

$$\int_a^b \frac{1}{x}\,dx = \ln|x|\Big|_a^b = \ln(|b|) - \ln(|a|).$$

Ex. What we defined as the integral $\int_a^b f(x)\,dx$ is in some places called a definite integral.

We proved in MATH 1052 that two functions with the same derivative can only differ by a constant (corollary 29.5 from the MVT).

Thus, if $F(x)$ is an antiderivative of $f(x)$, then so is $F(x)+c$ for any constant c, and furthermore, these are the only antiderivatives of f.

$F(x)+c$ is called the indefinite integral of $f(x)$, denoted $\int f(x)\,dx$.

Ex.
1. $\int \frac{1}{x}\,dx = \ln|x| + c.$ (example above)
2. $\int x^2\,dx = \frac{1}{3}x^3 + c.$ (example above)
3. $\int (2x^3 - 7x + 3\,dx = 2\int x^3 - 7\int x + \int 3\,dx$
   $= 2(\frac{1}{4}x^4) - 7(\frac{1}{2}x^2) + 3x + c.$
4. $\int e^x\,dx = e^x + c$ (since $(e^x)' = e^x$)
5. $\int e^{2x}\,dx = \frac{1}{2}e^{2x} + c$
6. $\int e^{-x^2}\,dx = \ldots$ $\quad -\frac{1}{2x}e^{x^2} + c$ (no obvious antiderivative)
7. Since $(b^x)' = b^x \ln b$,
   $\int b^x\,dx = \frac{1}{\ln b}b^x + c$
8. $\int \sin x\,dx = -\cos x + c$ (since $(-\cos x)' = \sin x$)
9. $\int \cos x\,dx = \sin x + c$
10. Find area under $\sin x$ between 0 and π.
    $$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 2.$$
    Between 0 and 2π.
    $$\int_0^{2\pi} \sin x\,dx = -\cos x\Big|_0^{2\pi} = -\cos(2\pi) - (-\cos 0) = 0.$$

Ex. For many functions, it can be difficult to find an antiderivative by inspection.

$\int \ln x \, dx$ — parts

$\int \frac{1}{x^2-1} \, dx$ — partial fractions

$\int \tan x \, dx$ — substitution

These techniques will be explored soon.

Thm. 34.2: Integration by Parts

Suppose u and v are continuous on $[a,b]$ and differentiable on (a,b), and suppose u' and v' are integrable on $[a,b]$. Then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\,dx.$$

Note: this is sometimes written as $\int u\,dv = uv - \int v\,du$.

Proof:

Let $g(x) = u(x)v(x)$, so that $g'(x) = u'(x)v(x) + u(x)v'(x)$ (product rule)

Since u and v are continuous, by thm. 33.2 they are integrable. Since by assumption u' and v' are integrable, by assignment 2 we have $u'v$ and uv' integrable.

Thm. 33.3 implies their sum is integrable, so g' is integrable. By the FTC, $\int_a^b g'(x)dx = g(x)\Big|_a^b = u(x)v(x)\Big|_a^b$.

By thm. 33.3,

$$\begin{aligned}\int_a^b g'(x)\,dx &= \int_a^b (u'(x)v(x) + u(x)v'(x))\,dx \\ &= \int_a^b u'(x)v(x)\,dx + \int_a^b u(x)v'(x)\,dx\end{aligned}$$

Thus $\int_a^b u'(x)v(x)\,dx + \int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b$. □

Ex. $\int_0^\pi x \sin x \, dx$

set $u(x) = x$, $u'(x) = 1$. set $v'(x) = \sin x$, $v(x) = -\cos x$.

$$\begin{aligned}\Rightarrow \int_0^\pi x\sin x\,dx &= -x\cos x\Big|_0^\pi - \int_0^\pi -\cos x\,dx \\ &= -\pi(-1) + \int_0^\pi \cos x\,dx \\ &= \pi + \sin x\Big|_0^\pi \\ &= \pi + \sin\pi - \sin 0 \\ &= \pi.\end{aligned}$$

We can use integration by parts for indefinite integrals: $\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx$.

$$\begin{aligned}\text{So } \int x\sin x\,dx &= -x\cos x - \int -\cos x\,dx \\ &= -x\cos x + \sin x + C.\end{aligned}$$

Ex. Find $\int \ln x \, dx$.

set $u(x) = \ln x$, $u'(x) = \frac{1}{x}$. set $v'(x) = 1$, $v(x) = x$.

$$\begin{aligned}\Rightarrow \int \ln x\,dx &= x\ln x - \int \left(\tfrac{1}{x}\right)(x)\,dx \\ &= x\ln x - \int 1\,dx \\ &= x\ln x - x + C\end{aligned}$$

$$\begin{aligned}\Rightarrow \int_1^e \ln x\,dx = (x\ln x - x)\Big|_1^e &= e\ln e - e - (\ln 1 - 1) \\ &= e - e - (-1) \\ &= 1.\end{aligned}$$

Ex. Find ① $\int xe^x\,dx$

② $\int \arctan(x)\,dx$

③ $\int \frac{1}{x^2+1}\,dx$

④ $\int (\ln x)^2\,dx$

Solution:

① $\int xe^x\,dx$

set $u = x$, $u' = 1$. set $v' = e^x$, $v = e^x$.

$\Rightarrow \int xe^x\,dx = xe^x - \int e^x\,dx$
$= xe^x - e^x + c$

② $\int \arctan x\,dx$

set $u = \arctan x$, $u' = \frac{1}{x^2+1}$. set $v' = 1$, $v = x$.

$\Rightarrow \int \arctan x\,dx = x\arctan x - \int \frac{x}{x^2+1}\,dx$ (note: $(\ln(x^2+1))' = \frac{2x}{x^2+1}$ $\Rightarrow \int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + c$.)
$= x\arctan x - \frac{1}{2}\ln(x^2+1) + c$

③ $\int \frac{1}{x^2+1}\,dx = \arctan x + c$

④ $\int (\ln x)^2\,dx$

set $u = (\ln x)^2$, $u' = 2\ln x\left(\frac{1}{x}\right)$. set $v' = 1$, $v = x$.

$\Rightarrow \int (\ln x)^2\,dx = x(\ln x)^2 - 2\int \ln x\,dx = x(\ln x)^2 - 2(x\ln x - x) + c$
$= x((\ln x)^2 - 2\ln x + 2) + c$

Ex. Find $\int x^2\cos(2x)\,dx$

set $u_1 = x^2$, $u_1' = 2x$. set $v_1' = \sin(2x)$, $v_1 = -\frac{1}{2}\cos(2x)$

$\Rightarrow \int x^2\cos(2x)\,dx = \frac{x^2}{2}\sin(2x) - \int x\sin(2x)\,dx$

set $u_2 = x$, $u_2' = 1$. set $v_2' = \sin(2x)$, $v_2 = \frac{1}{2}\cos(2x)$

$\Rightarrow \int x\sin(2x)$