Note: Under the assumptions of the FTC and taking F with $F' = f$,

$$\begin{aligned}\int_a^b f(x)\,dx &= F(b) - F(a) \\ &= -(F(a) - F(b)) \\ &= -\int_b^a f(x)\,dx\end{aligned}$$

We only defined $\int_a^b f$ if $a < b$, but because of the above, we can extend our definition.

Def.

If $b < a$, then $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

One can check that all previous theorems still work as stated.

Ex. Given this, we have

$$\begin{aligned}\int_a^a f(x)\,dx &= \int_a^b f(x)\,dx + \int_b^a f(x)\,dx \\ &= \int_a^b f(x)\,dx - \int_a^b f(x)\,dx \\ &= 0.\end{aligned}$$

Thm. 34.3: Fundamental Theorem of Calculus II

Let f be integrable on $[a,b]$.

For $x \in [a,b]$, let $F(x) = \int_a^x f(t)\,dt$.

Then F is continuous on $[a,b]$.

If f is continuous at $x_0 \in (a,b)$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Alternative notation: $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

Proof:

Since f is integrable on $[a,b]$, it is bounded on $[a,b]$.

So there exists $B > 0$ such that $|f(x)| \le B \;\forall x \in [a,b]$.

Now let $\varepsilon > 0$ be given and suppose $x, y \in [a,b]$ with $|x-y| < \frac{\varepsilon}{B}$.

Without loss of generality, suppose $x < y$. Then

$$\begin{aligned}|F(y) - F(x)| &= \left|\int_a^y f(t)\,dt - \int_a^x f(t)\,dt\right| \\ &= \left|\int_a^y f(t)\,dt + \int_x^a f(t)\,dt\right| \\ &= \left|\int_x^y f(t)\,dt\right| && \text{(by theorem 33.6)} \\ &\le \int_x^y |f(t)|\,dt && \text{(by theorem 33.5)} \\ &\le \int_x^y B\,dt && \text{(by theorem 33.4 (1))} \\ &= B(y-x) \\ &< B\left(\frac{\varepsilon}{B}\right) && \left(\text{since } y-x = |y-x| = |x-y| < \frac{\varepsilon}{B}\right) \\ &= \varepsilon.\end{aligned}$$

Thus F is uniformly continuous on $[a,b]$.

Now suppose f is continuous at $x_0 \in (a,b)$.

For $x \neq x_0$, we have

$$F(x) - F(x_0) = \int_a^x f(t)\,dt - \int_a^{x_0} f(t)\,dt$$
$$= \int_a^x f(t)\,dt + \int_{x_0}^a f(t)\,dt$$
$$= \int_{x_0}^x f(t)\,dt,$$

and so $\frac{F(x)-F(x_0)}{x-x_0} = \frac{1}{x-x_0}\int_{x_0}^x f(t)\,dt$.

We also have

$$\frac{1}{x-x_0}\int_{x_0}^x f(x_0)\,dt = \frac{f(x_0)}{x-x_0}\int_{x_0}^x 1\,dt$$
$$= \frac{f(x_0)}{x-x_0}(x-x_0)$$
$$= f(x_0).$$

Thus $\frac{F(x)-F(x_0)}{x-x_0} - F(x_0) = \frac{1}{x-x_0}\int_{x_0}^x (f(t)-f(x_0))\,dt$ ⊛

(Goal: show the limit as $x \to x_0$ exists and equals 0).

Let $\varepsilon > 0$ be given. Since f is continuous at x_0, there is a $\delta > 0$ such that $t \in (a,b)$ and $|t - x_0| < \delta$ imply $|f(t) - f(x_0)| < \varepsilon$. (limit definition of continuity at x_0).

In the case $x > x_0$, ⊛ with the above bound yields:

$$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| = \frac{1}{|x-x_0|}\left|\int_{x_0}^x (f(t)-f(x_0))\,dt\right|$$
$$\leq \frac{1}{x-x_0}\int_{x_0}^x |f(t)-f(x_0)|\,dt \quad \text{(by theorem 33.5)}$$
$$\leq \frac{1}{x-x_0}\int_{x_0}^x \varepsilon\,dt \quad \text{(by theorem 33.4 (ii))}$$
$$= \frac{1}{x-x_0}\varepsilon(x-x_0)$$
$$= \varepsilon.$$

In the case $x < x_0$, ⊛ yields:

$$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| = \frac{1}{|x-x_0|}\left|\int_{x_0}^x (f(t)-f(x_0))\,dt\right|$$
$$= \frac{1}{-(x-x_0)}\left|\int_x^{x_0} (f(t)-f(x_0))\,dt\right|$$
$$\leq \frac{1}{-(x-x_0)}\varepsilon(x_0-x)$$
$$= \varepsilon.$$

By the definition of a limit, we've shown that

$$\lim_{x\to x_0}\frac{F(x)-F(x_0)}{x-x_0} = f(x_0),$$

which implies $F'(x_0) = f(x_0)$. □

Ex. Let $g(x) = \int_1^x t^2\,dt$. Then $g'(x) = x^2$ by FTC II.

To check: $\int_1^x t^2\,dt = \frac{1}{3}t^3\Big|_1^x$ (by FTC I)

$$= \frac{1}{3}x^3 - \frac{1}{3}$$

$$\Rightarrow \left(\frac{1}{3}x^3 - \frac{1}{3}\right)' = x^2 = g'(x).$$

Ex. Let $p(x) = \int_0^x e^{-t^2}\,dt$.

Then $p'(x) = e^{-x^2}$.

Ex. Let $F(x) = \int_{2x}^{x^2} e^{-t^2}\,dt$.

Let $p(x) = \int_0^x e^{-t^2}\,dt$ so that $p'(x) = e^{-x^2}$.

Then $F(x) = \int_{2x}^{0} e^{-t^2}\,dt + \int_0^{x^2} e^{-t^2}\,dt$

$= -\int_0^{2x} e^{-t^2}\,dt + \int_0^{x^2} e^{-t^2}\,dt$

$= -p(2x) + p(x^2)$ ← compositions

Thus by the chain rule:

$F'(x) = -p'(2x)(2) + p'(x^2)(2x)$

$= -2e^{-(2x)^2} + 2xe^{-(x^2)^2}$

$= -2e^{-4x^2} + 2xe^{-x^4}$.

Ex. Let $Li(x) = \int_2^x \frac{1}{\ln t}\,dt$.

Then $Li'(x) = \frac{1}{\ln x}$

Logarithmic integral function. (Important in number theory)

Extra (non-course) material:

Let $\pi(x) = \sum_{\substack{p \le x \\ p \text{ prime}}} 1$

(prime counting function)

Prime Number Theorem: 1896

$\lim_{x\to\infty} \frac{\pi(x)}{Li(x)} = 1$. (ie. $\pi(x) \sim Li(x)$)

(Note: $Li(x) \sim \frac{x}{\ln x}$, so $\pi(x) \sim \frac{x}{\ln x}$.)

$\frac{\pi(x)}{x} \sim \frac{1}{\ln x}$

Riemann hypothesis: (open problem)

$|\pi(x) - Li(x)| \le c\sqrt{x}\ln x$ for some c, once x gets big enough.