Thm 34.4: Change of variables / substitution

Let u be a differentiable function on an open interval J with u' continuous, and let I be an open interval with $u(x) \in I$ $\forall x \in I$.

If f is continuous on I, then $f \circ u$ is continuous on J with

$$\int_a^b (f \circ u)(x)\, u'(x)\, dx = \int_{u(a)}^{u(b)} f(u)\, du$$

for $a, b \in J$.

Proof:

$f \circ u$ is continuous by thm. 17.5 (MATH 1052).

Let $c \in I$ and define $F(u) = \int_c^u f(t)\, dt$.

By FTC II, $F'(u) = f(u)$ $\forall u \in I$.

Let $g = F \circ u$. By the chain rule,

$$g'(x) = F'(u(x))\, u'(x) = f(u(x))\, u'(x).$$

Hence
$$\begin{aligned}
\int_a^b (f \circ u)(x)\, u'(x)\, dx &= \int_a^b g'(x)\, dx \\
&= g(b) - g(a) \quad \text{(by FTC II)} \\
&= F(u(b)) - F(u(a)) \\
&= \int_c^{u(b)} f(t)\, dt - \int_c^{u(a)} f(t)\, dt \\
&= \int_c^{u(b)} f(t)\, dt + \int_{u(a)}^{c} f(t)\, dt \\
&= \int_{u(a)}^{u(b)} f(t)\, dt. \qquad \square
\end{aligned}$$

Ex. To use thm. 34.4, look for integrals of the form

$$\int f(u(x))\, u'(x)\, dx.$$

Ex. $\int_0^2 x e^{-x^2}\, dx$.

Let $u(x) = -x^2$. Then $u'(x) = -2x$, and so $f(x) = -\frac{1}{2} e^x$.

$\Rightarrow f(u(x))\, u'(x) = \left(-\frac{1}{2} e^{-x^2}\right)(-2x) = x e^{-x^2}$.

$u(0) = 0$, $u(2) = -4$.

Thus
$$\begin{aligned}
\int_0^2 x e^{-x^2}\, dx &= \int_0^{-4} -\tfrac{1}{2} e^u\, du \\
&= \tfrac{1}{2} \int_{-4}^{0} e^u\, du \\
&= \tfrac{1}{2}\left(e^{-4} - e^0\right) \\
&= \tfrac{1}{2} e^{-4} - \tfrac{1}{2}.
\end{aligned}$$

Alternatively:

$u = -x^2$ $\qquad u(0) = 0$

$du = -2x\, dx$ $\qquad u(2) = -4$

So $x\,dx = -\frac{1}{2}\, du$

$\Rightarrow -\frac{1}{2} \int_0^{-4} e^u\, du$.

Ex. $\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx$

$u = x^3+1,\ du = 3x^2 \Rightarrow \frac{1}{3}du = x^2\,dx$

$u(0)=1,\ u(2)=9$

$$\Rightarrow \frac{1}{3}\int_1^9 \frac{du}{\sqrt{u}} = \frac{1}{3}\int_1^9 u^{-\frac{1}{2}}\,du$$
$$= \frac{2}{3}u^{\frac{1}{2}}\Big|_1^9$$
$$= \frac{2}{3}(3-1) = \frac{4}{3}.$$

Ex. We can use substitution on indefinite integrals as well.

$\int x e^{-x^2}\,dx$

$u = -x^2,\ du = -2x\,dx \Rightarrow -\frac{1}{2}du = x\,dx$

$$\Rightarrow -\frac{1}{2}\int e^u\,du = -\frac{1}{2}e^u + c$$
$$= -\frac{1}{2}e^{-x^2} + c. \quad (\text{since } u = -x^2.)$$

Ex. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx.$

$u = \cos x,\ du = -\sin x\,dx \Rightarrow -du = \sin x\,dx$

$$\Rightarrow -\int \frac{1}{u}\,du = -\ln|u| + c$$
$$= -\ln|\cos x| + c$$
$$= \ln|\cos x|^{-1} + c \quad (\text{log law})$$
$$= \ln|\cos^{-1} x| + c$$
$$= \ln|\sec x| + c \quad \left(\sec x = \frac{1}{\cos x}\right).$$

Ex. $\int x\sqrt{4-x}\,dx.$

$u = 4-x,\ du = -dx.$ Also, $x = 4-u.$

$$\Rightarrow -\int (4-u)\sqrt{u}\,du = -\int (4\sqrt{u} - u\sqrt{u})\,du$$
$$= -\int \left(4u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$
$$= -\left(\frac{8}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right) + c$$
$$= -\frac{8}{3}u^{\frac{3}{2}} + \frac{2}{5}u^{\frac{5}{2}} + c$$
$$= -\frac{8}{3}(4-x)^{\frac{3}{2}} + \frac{2}{5}(4-x)^{\frac{5}{2}} + c.$$

Ex. $\int \frac{x+1}{x^2+1}\,dx = \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx$ (second integral is $\arctan x$ by FTC)

In first integral, let $u = x^2+1$, so $du = 2x\,dx.$

$$\Rightarrow \frac{1}{2}\int \frac{1}{u}\,du + \arctan x + c = \frac{1}{2}\ln|u| + \arctan x + c$$
$$= \frac{1}{2}\ln|x^2+1| + \arctan x + c$$
$$= \frac{1}{2}\ln(x^2+1) + \arctan x + c.$$

Ex. $\int \frac{1}{1+e^x}\,dx$

$u = 1+e^x$, $du = e^x\,dx \Rightarrow dx = \frac{1}{e^x}\,du = \frac{1}{u-1}\,du$

But this substitution is harmful, since we don't know yet how to integrate $\int \frac{1}{u(u-1)}\,du$.

Instead, change the integrand.

$\int \frac{1}{1+e^x}\,dx = \int \frac{e^{-x}}{e^{-x}+1}\,dx$ (multiply by $\frac{e^{-x}}{e^{-x}}$)

$u = e^{-x}+1$, $du = -e^{-x}\,dx$

$\Rightarrow -\int \frac{1}{u}\,du = -\ln|u| + c$

$= -\ln(e^{-x}+1) + c.$

Ex. $\int \sec x\,dx$

Note: $(\sec x)' = \sec x \tan x$

$(\tan x)' = \sec^2 x$

So: $(\sec x + \tan x)' = \sec x(\sec x + \tan x)$.

$\Rightarrow \int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx.$

$u = \sec x + \tan x$, $du = \sec x(\sec x + \tan x)\,dx$

$\Rightarrow \int \frac{1}{u}\,du = \ln|u| + c$

$= \ln|\sec x + \tan x| + c.$

A question like this would require pretty big hints on a test/assignment.

Ex. For comparison, $\int \sec^2 x = \tan x + c$ (by FTC).

Ex. $\int \sec^3 x\,dx$

Note: $\sec^3 x = (\sec x)(\sec^2 x)$

Using integration by parts:

$\int u\,dv = uv - \int v\,du$

$u = \sec x$, $du = \sec x \tan x\,dx$

$dv = \sec^2 x\,dx$, $v = \tan x$

$\Rightarrow \sec x \tan x - \int \sec x \tan^2 x\,dx = \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$ (trig identity)

$= \sec x \tan x - \int \sec^3 dx + \int \sec x\,dx.$

Add $\int \sec^3 dx$ to both sides.

$\Rightarrow 2\int \sec^3 dx = \sec x \tan x + \int \sec x\,dx$

$= \sec x \tan x + \ln|\sec x + \tan x| + c$

$\Rightarrow \int \sec^3 dx = \frac{1}{2}(\sec x \tan x) + \frac{1}{2}\ln|\sec x \tan x| + c.$

Ex. $\int \cos(\ln x)\,dx$

Let $t = \ln x$, $dt = \frac{1}{x}dx \Rightarrow dx = x\,dt = e^t dt$ (since $t = \ln x \Rightarrow e^t = x$)

$\Rightarrow \int e^t \cos(t)\,dt$

This is on assignment 3.

Hint from class: use integration by parts twice, and the integral will appear on both sides like last example.

$\Rightarrow \int e^t \cos(t)\,dt = \frac{1}{2}e^t(\cos t + \sin t) + c$

$= \frac{1}{2}x\left(\cos(\ln x) + \sin(\ln x)\right) + c$

Ex. $\int x^5 e^{x^2}\,dx$

Let $t = x^2$, $dt = 2x\,dx \Rightarrow \frac{1}{2}dt = x\,dx$ $(x^4 = t^2)$

$\Rightarrow \int x^4 x e^{x^2}\,dx = \frac{1}{2}\int t^2 e^t\,dt$

use integration by parts

$u = t^2$, $du = 2t$

$dv = e^t$, $v = e^t$

$\Rightarrow \frac{1}{2}\left(t^2e^t - 2\int te^t\,dt\right) = \frac{1}{2}t^2e^t - \int te^t\,dt$

use integration by parts again

$u = t$, $du = dt$

$dv = e^t dt$, $v = e^t$

$\Rightarrow \frac{1}{2}t^2e^t - \left(te^t - \int e^t\,dt\right) = \frac{1}{2}t^2e^t - te^t + e^t + c$

$= \frac{1}{2}e^t\left(t^2 - 2t + 2\right) + c$

$= \frac{1}{2}e^{x^2}\left(x^4 - 2x^2 + 2\right) + c.$