Note We sometimes need to integrate functions that have a factor of the form $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ or $\sqrt{x^2-a^2}$, for some constant a.

Recall that $\sin^2\theta + \cos^2\theta = 1$, so $1-\sin^2\theta = \cos^2\theta$, $\tan^2\theta + 1 = \sec^2\theta$, and $\sec^2\theta - 1 = \tan^2\theta$.

We make substitutions: $x = a\sin\theta$, $x = a\tan\theta$, $x = a\sec\theta$ to get rid of the square root.

Ex. $\int \frac{1}{\sqrt{1-x^2}}\,dx$.

Let $x = \sin\theta$, so $dx = \cos\theta\, d\theta$.

$1-x^2 = 1-\sin^2\theta = \cos^2\theta$.

Note that $\sqrt{1-x^2} = \sqrt{\cos^2\theta} = |\cos\theta|$, but for $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, $\cos\theta > 0$, so $|\cos\theta| = \cos\theta$.

For these θ, we have $-1 < x < 1$, which covers all x in the domain of the integrand.

$$\Rightarrow \int \frac{1}{\cos\theta}\cos\theta\, d\theta = \int 1\, d\theta$$
$$= \theta + c$$
$$= \arcsin x + c \qquad (\text{since } x = \sin\theta \Rightarrow \theta = \arcsin x)$$

Ex. $\int_2^{2\sqrt{2}} \sqrt{x^2-4}\,dx$

Let $x = 2\sec\theta$, so $dx = 2\sec\theta\tan\theta\, d\theta$.

So $x^2 - 4 = 4\sec^2\theta - 4$
$= 4(\sec^2\theta - 1)$
$= 4\tan^2\theta$.

$\sec\theta = \frac{x}{2} \Rightarrow \frac{1}{\cos\theta} = \frac{x}{2} \Rightarrow \cos\theta = \frac{2}{x}$.

When $x = 2$, $\cos\theta = 1$, so $\theta = 0$.

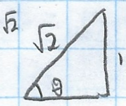
When $x = 2\sqrt{2}$, $\cos\theta = \frac{1}{\sqrt{2}}$, so $\theta = \frac{\pi}{4}$.

$$\Rightarrow \int_0^{\pi/4} \sqrt{4\tan^2\theta}\cdot 2\sec\theta\tan\theta\, d\theta = 4\int_0^{\frac{\pi}{4}} \sec\theta\tan^2\theta\, d\theta$$
$$= 4\int_0^{\pi/4} \sec\theta(\sec^2\theta - 1)\, d\theta$$
$$= 4\int_0^{\pi/4} \sec^3\theta - \sec\theta\, d\theta$$
$$= 4\left(\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta+\tan\theta| - \ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4} \quad (\text{from past examples})$$
$$= 2(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|)\Big|_0^{\pi/4} \quad \circledast$$
$$= 2(\sqrt{2} - \ln|\sqrt{2}+1| - (0 - \ln|1|))$$
$$= 2(\sqrt{2} - \ln(\sqrt{2}+1)).$$

$\sec\frac{\pi}{4} = \frac{1}{\cos\frac{\pi}{4}} = \sqrt{2}$

$\sec 0 = \frac{1}{\cos 0} = 1$

$\tan\frac{\pi}{4} = 1$



Say we want $\int\sqrt{x^2-4}\,dx$. By $\circledast$, we'd get to
$2(\sec\theta\tan\theta - \ln|\sec\theta + \tan\theta|) + c$.

We have $\sec\theta = \frac{x}{2}$, so $\tan^2\theta = \sec^2\theta - 1 = \frac{x^2}{4} - 1 = \frac{1}{4}(x^2-4)$, so $\tan\theta = \frac{\sqrt{x^2-4}}{2}$.

So the integral is

$$2\left(\frac{x}{2}\frac{\sqrt{x^2-4}}{2} - \ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right|\right) + c = \frac{x}{2}\sqrt{x^2-4} - \ln\left|\frac{x+\sqrt{x^2-4}}{2}\right| + c$$
$$= \frac{x}{2}\sqrt{x^2-4} - \left(\ln|x+\sqrt{x^2-4}| - \ln 2\right) + c$$
$$= \frac{x}{2}\sqrt{x^2-4} - \ln|x+\sqrt{x^2-4}| + c.$$

Ex. Finding the radius of a circle.

$x^2 + y^2 = r^2 \Rightarrow y^2 = r^2 - x^2$

$\Rightarrow y = \pm\sqrt{r^2 - x^2}$.

Integrate $\sqrt{r^2 - x^2}$ from 0 to r, then multiply by 4.

$\text{Area} = 4\int_0^r \sqrt{r^2 - x^2}\,dx$.

Let $x = r\sin\theta$, $dx = r\cos\theta\,d\theta$

$r^2 - x^2 = r^2 - r^2\sin^2\theta$

$= r^2(1 - \sin^2\theta)$

$= r^2\cos^2\theta$

$\sin\theta = \frac{x}{r}$

When $x = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0$.

When $x = r \Rightarrow \sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$.

$$\Rightarrow 4\int_0^{\pi/2} \sqrt{r^2\cos^2\theta}\cdot r\cos\theta\,d\theta = 4r^2\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 4r^2\int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\,d\theta$$

$$= 2r^2\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2}$$

$$= 2r^2\left(\frac{\pi}{2} + 0 - 0\right)$$

$= \pi r^2$, as expected.

Ex. $\int \frac{x^3}{(4x^2+9)^{3/2}}\,dx$

Let $x = \frac{3}{2}\tan\theta \Rightarrow dx = \frac{3}{2}\sec^2\theta\,d\theta$

$4x^2 + 9 = 9\tan^2\theta + 9$

$= 9(\tan^2\theta + 1)$

$= 9\sec^2\theta$.

$$\Rightarrow \int \frac{(27/8)\tan^3\theta}{(9\sec^2\theta)^{3/2}}\cdot\frac{3}{2}\sec^2\theta\,d\theta = \frac{3}{16}\int \frac{\tan^3\theta}{\sec\theta}\,d\theta$$ (simplify/combine terms)

$\frac{\tan^3\theta}{\sec\theta} = \frac{\sin^3\theta}{\cos^3\theta}\cos\theta$

$$= \frac{3}{16}\int \frac{\sin^3\theta}{\cos^2\theta}\,d\theta$$

$$= \frac{3}{16}\int \frac{(1-\cos^2\theta)\sin\theta}{\cos^2\theta}\,d\theta$$

Let $u = \cos\theta$, $du = -\sin\theta\,d\theta$.

$$= -\frac{3}{16}\int \frac{1-u^2}{u^2}\,du$$

$$= -\frac{3}{16}\int \left(\frac{1}{u^2} - 1\right)du$$

$$= -\frac{3}{16}\left(-\frac{1}{u} - u\right) + C$$

$$= \frac{3}{16}\,\frac{u^2+1}{u} + C$$

$$= \frac{3}{16}\,\frac{\cos^2\theta + 1}{\cos\theta} + C$$

but $x = \frac{3}{2}\tan\theta \Rightarrow \tan\theta = \frac{2}{3}x$

$\sec^2\theta = \tan^2\theta + 1 = \frac{4x^2}{9} + 1 = \frac{4x^2+9}{9}$

so $\cos^2\theta = \frac{1}{\sec^2\theta} = \frac{9}{4x^2+9}$

$\Rightarrow \cos\theta = \frac{3}{\sqrt{4x^2+9}}$

$$= \frac{3}{16}\,\frac{\frac{9}{4x^2+9} + 1}{\sqrt{\ }} + C$$

$=$ (simplify)

$$= \frac{2x^2+9}{8\sqrt{4x^2+9}} + C$$

Partial Fractions

Ex. Find $\int \frac{1}{x^2-1}\,dx$ (recall: $\int \frac{1}{x^2+1}\,dx = \arctan x + c$)

Note: $x^2-1 = (x-1)(x+1)$

Find A and B such that

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}$$

$\Rightarrow 1 = A(x+1) + B(x-1)$ (multiply by $(x+1)(x-1)$)

must hold for all x.

Either equate coefficients of like powers of x or use special values.

option 1: $(A+B)x + (A-B) = 1$

$A+B = 0 \Rightarrow A = \frac{1}{2}$

$A-B = 1 \quad B = -\frac{1}{2}$

option 2:

let $x=1$: $2A = 1 \Rightarrow A = \frac{1}{2}$

let $x=-1$: $-2B = 1 \Rightarrow B = -\frac{1}{2}$

Thus $\int \frac{1}{x^2-1}\,dx = \int \left(\frac{1/2}{x-1} - \frac{1/2}{x+1}\right) dx$

$= \frac{1}{2}\int \frac{1}{x-1}\,dx - \frac{1}{2}\int \frac{1}{x+1}\,dx$

$= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + c$

$= \ln\left(\sqrt{\left|\frac{x-1}{x+1}\right|}\right) + c.$

Partial Fraction Decomposition

Note Consider $\frac{f(x)}{g(x)}$ for polyno

If $\deg(f) \geq \deg(g)$, we may use polynomial long division for equations to write

$\frac{f(x)}{g(x)} = h(x) + \frac{f_1(x)}{g(x)}$

Where h is a polynomial and $\deg(f_1) < \deg(g)$

We assume $\deg(f) < \deg(g)$. Suppose g factors (over $\mathbb{R}$) as $g(x) = p_1(x)^{n_1} \cdot p_2(x)^{n_2} \cdots p_k(x)^{n_k}$

for coprime irreducible polynomials

We may then write $\frac{f(x)}{g(x)} = \sum_{i=1}^{k} T_i(x)$ where $T_i(x)$ is as follows.

If $\deg(p_i) = 1$, then $T_i(x) = \frac{A_1}{p_i(x)} + \frac{A_2}{p_i(x)^2} + \cdots + \frac{A_{n_i}}{p_i(x)^{n_i}}$, for $A_i \in \mathbb{R}$.

If $\deg(p_i) = 2$, then $T_i(x) = \frac{A_1x+B_1}{p_i(x)} + \frac{A_2x+B_2}{p_i(x)^2} + \cdots + \frac{A_{n_i}x+B_{n_i}}{p_i(x)^{n_i}}$, for $A_i, B_i \in \mathbb{R}$

Proof of existence and uniqueness is omitted.

Ex. $\frac{2x+1}{x^2+x-2} = \frac{2x+1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$

Multiply by denominator

$\Rightarrow A(x+2) + B(x-1) = 2x+1$

Let $x=1 \Rightarrow 3A$

$x=-2 \Rightarrow -3B = -3 \Rightarrow B = 1$

So $\frac{2x+1}{x^2+x-2} = \frac{1}{x-1} + \frac{1}{x+2}$. Thus $\int \frac{2x+1}{x^2+x-2}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{1}{x+2}\,dx$

$= \ln|x-1| + \ln|x+2| + c.$