Definition of Vector Spaces

Ex.

Usually, $F = \mathbb{R}$ or $\mathbb{C}$.

Some examples of vector spaces:

$F^n = \left\{ \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} : c_1, \dots, c_n \in F \right\}$

$\Rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} c_1+d_1 \\ \vdots \\ c_n+d_n \end{bmatrix} \quad \forall \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} \in V$

$\Rightarrow \lambda \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} \lambda c_1 \\ \vdots \\ \lambda c_n \end{bmatrix} \quad \forall \lambda \in F, \forall \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \in V$

$\mathbb{R}^\infty = \{(c_1, c_2, c_3, \dots) : c_1, c_2, c_3, \dots \in \mathbb{R}\}$ is the set of real sequences.

$\Rightarrow$ Define $(c_1, \dots) + (d_1, \dots) = (c_1 + d_1, \dots)$ as addition

$\Rightarrow$ Define $\lambda(c_1 \dots) = (\lambda c_1, \dots)$ as scalar multiplication

Exercise: show $\mathbb{R}^\infty$ is a vector space.

Proof: commutativity of addition in $\mathbb{R}^\infty$

$$\begin{aligned} (c_1, c_2, \dots) + (d_1, d_2, \dots) &= (c_1+d_1, c_2+d_2, \dots) \quad && \forall (c_1, c_2, \dots), (d_1, d_2, \dots) \in \mathbb{R}^\infty \\ &= (d_1+c_1, d_2+c_2, \dots) && \text{(since addition is commutative in } \mathbb{R}) \\ &= (d_1, d_2, \dots) + (c_1, c_2, \dots). \quad \square \end{aligned}$$

$\mathbb{C}^\infty$ and $\mathbb{R}^\infty$ are also vector spaces.

Ex.

Suppose S is a nonempty set. Define

$\mathbb{F}^S$ = set of all functions $f: S \to \mathbb{F}$.

For example, $\mathbb{R}^\mathbb{R}$ = set of all real valued functions on $\mathbb{R}$.

Define addition in $\mathbb{F}^S$ by

$(f+g)(s) = f(s) + g(s) \quad \forall s \in S, \forall f, g \in \mathbb{F}^S$.

Define scalar multiplication in $\mathbb{F}^S$ by

$(\lambda f)(s) = \lambda(f(s)) \quad \forall s \in S, \forall \lambda \in \mathbb{F}, f \in \mathbb{F}^S$.

Define $z: S \to \mathbb{F}$ by $z(s) = 0 \quad \forall s \in S$.

To show z is the additive identity in $\mathbb{F}^S$, must verify $f + z = f \quad \forall f \in \mathbb{F}^S$.

ie. $(f+z)(s) = f(s) \quad \forall s \in S$.

Proof:

$$\begin{aligned} (f+z)(s) &= f(s) + z(s) \\ &= f(s) + 0 \\ &= f(s). \quad \square \end{aligned}$$

Exercise: show $\mathbb{F}^S$ is a vector space.

Ex.

Take $S = [0,1] \subseteq \mathbb{R}$, $F = \mathbb{R}$. Then

$\mathbb{R}^{[0,1]}$ = set of functions from $[0,1]$ to $\mathbb{R}$.

Let $f(s) = \cos(s)$ $\forall s \in [0,1]$

$g(s) = s^2+1$ $\forall s \in [0,1]$

Then $f, g \in \mathbb{R}^{[0,1]}$, and $(f+g)(s) = \cos(s) + s^2 + 1$.

Ex.

Take $S = \{1,2\}$ and $\mathbb{F} = \mathbb{R}$.

Then $\mathbb{R}^{\{1,2\}}$ = functions $f: \{1,2\} \to \mathbb{R}$.

Suppose $f \in \mathbb{R}^{\{1,2\}}$. Then we get an element in $\mathbb{R}^2$ given by $\begin{bmatrix} f(1) \\ f(2) \end{bmatrix} \in \mathbb{R}^2$.

On the other hand, given $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \in \mathbb{R}^2$, define $g: \{1,2\} \to \mathbb{R}$ by $g(1) = c_1$ and $g(2) = c_2$.

The two maps $f \mapsto \begin{bmatrix} f(1) \\ f(2) \end{bmatrix}$ and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \mapsto g$ are inverses of each other, and so the two maps define bijections between $\mathbb{R}^2$ and $\mathbb{R}^{\{1,2\}}$.

In fact, these maps are isomorphisms.

Exercise

$\mathbb{F}^n \cong \mathbb{F}^{\{1,\dots,n\}}$.

Thm. 1.26

Suppose V is a vector space over F.

V has a unique additive identity.

Proof:

Suppose $0, 0'$ are additive identities, e.g.

$v + 0 = v$ and $v + 0' = v$. $\forall v \in V$.

$\Rightarrow 0 = 0 + 0'$ (by $v + 0' = v$)

$= 0' + 0$ (by commutativity)

$= 0'$ (by $v + 0 = v$). $\square$

Thm 1.27

Every $v \in V$ has a unique additive inverse.

Sketch of proof:

Suppose w and w' are additive inverses to $v \in V$

Then $v + w = 0 = v + w'$

$\Rightarrow v + w = v + w'$

$\Rightarrow w + (v+w) = w + (v+w')$

$\Rightarrow (w+v) + w = (w+v) + w'$

$\Rightarrow 0 + w = 0 + w'$

$\Rightarrow w = w'$.

Thm. 1.30

$0v = 0 \quad \forall v \in V.$

Proof omitted.

Thm. 1.31

$\lambda 0 = 0 \quad \forall \lambda \in \mathbb{F}.$

Proof omitted.

Thm. 1.32

$(-1)v = -v \quad \forall v \in V$

Proof omitted.

Subspaces (1C)

Def.

A subspace of V is a subset $W \subseteq V$ which is also a vector space using the same addition and scalar multiplication from V.

Thm. 1.34 (Subspace test)

$W \subseteq V$ is a subspace $\Leftrightarrow$ 1. $0 \in W$
2. $u, w \in W \Rightarrow u+w \in W$
3. $\lambda \in \mathbb{F},\ w \in W \Rightarrow \lambda w \in W.$

Proof omitted.

Ex.

$\left\{ \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} : ac_1 + bc_2 = 0 \right\}$ for fixed $a, b \in \mathbb{R}$ is a subspace of $\mathbb{R}^2$.

Ex.

Let W be the set of continuous functions in $\mathbb{R}^{\mathbb{R}}$ = functions from $\mathbb{R}$ to $\mathbb{R}$.
We know the zero function $z: \mathbb{R} \to \mathbb{R}$ is continuous (constant functions are continuous).
We know that $f+g$ is continuous if $f, g \in W$ and λf is continuous if $f \in W$ $\lambda \in \mathbb{R}$.