Def. 1.36

Suppose $W_1, \ldots, W_m \subseteq V$ are subspaces.
The sum of $W_1, \ldots, W_m$ is
$$W_1 + \ldots + W_m = \{w_1 + \ldots + w_m : w_1 \in W_1, \ldots, w_m \in W_m\}.$$

Ex.

Let $U = \{(a,0,0) : a \in \mathbb{R}\} \subseteq \mathbb{R}^3$ (a subspace) and $W = \{(0,b,0) : b \in \mathbb{R}\} \subseteq \mathbb{R}^3$ (a subspace).
$$U + W = \{(a,0,0) + (0,b,0) : a,b \in \mathbb{R}\}$$
$$= \{(a,b,0) : a,b \in \mathbb{R}\}.$$
Geometrically, this is the xy-plane (which is also a subspace).

Exercise:

Compute $U + W$ where $U = \{(a,a,0) : a \in \mathbb{R}\} \subseteq \mathbb{R}^3$ and $W = \{(b,b,b) : b \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

Thm. 1.40

Suppose $W_1, \ldots, W_m \subseteq V$ are subspaces. Then
$W_1 + \ldots + W_m$ is a subspace of V.

Sketch of proof: use the subspace test.

Def. 1.41

Suppose $W_1, \ldots, W_m \subseteq V$ are subspaces. Then
$$W_1 + \ldots + W_m$$
is called an (internal) direct sum if
$$w_1 + \ldots + w_m = w_1' + \ldots + w_m'$$
for $w_1, w_1' \in W_1, \ldots, w_m, w_m' \in W_m$, then $w_1 = w_1', \ldots, w_m = w_m'$.
ie. every vector in $W_1 + \ldots + W_m$ may be written uniquely as $w_1 + \ldots + w_m$ with $w_1 \in W_1, \ldots, w_m \in W_m$.
In this case, we write $W_1 \oplus \ldots \oplus W_m$.
Note: $\oplus$ is not an operation.

Ex.

Take $U = \mathbb{R}^2$ and $V = \mathbb{R}^2$.
Then $U + V = \{u + v : u, v \in \mathbb{R}^2\} = \mathbb{R}^2$.
Looking for $u + v = u' + v$ where $u, u' \in U$ and $v, v^2 \in V$.
Take $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $u' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $u = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
Then $u + v = v + w$, but $u \neq u'$ and $v \neq v'$.

Exercise:

Take $U = \{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \in \mathbb{R} \} \subseteq \mathbb{R}^2$ and $V = \mathbb{R}^2$.

Show that $U + V$ is not direct.

Ex.

Set $W_1 = \{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{F}\} \subseteq \mathbb{F}^3$,

$W_2 = \{(0, 0, a) : a \in \mathbb{F}\} \subseteq \mathbb{F}^3$,

$W_3 = \{(0, a, a) : a \in \mathbb{F}\} \subseteq \mathbb{F}^3$, which are subspaces.

This sum $W_1 + W_2 + W_3$ is not direct because

$$(0,-1,0) + (0,0,-1) + (0,1,1) = (0,0,0) \text{ and}$$
$$\underset{\in W_1}{(0,0,0)} + \underset{\in W_2}{(0,0,0)} + \underset{\in W_3}{(0,0,0)} = (0,0,0).$$

Since $(0,-1,0) \neq (0,0,0)$, $W_1 + W_2 + W_3$ is not a direct sum.

Thm. 1.45

Suppose $W_1, \dots, W_m \subseteq V$ are subspaces. Then

$W_1 + \dots + W_m = W_1 \oplus \dots \oplus W_m$

$\Leftrightarrow 0 \in W_1 + \dots + W_m$ may be written uniquely as $0 = 0 \in W_1 + \dots + 0 \in W_m$.

Proof:

"$\Rightarrow$" Suppose $W_1 + \dots + W_m = W_1 \oplus \dots \oplus W_m$, i.e. is a direct sum.

Then by definition every vector in $W_1 + \dots + W_m$ can be written uniquely as

$w_1 + \dots + w_m$, $w_1 \in W_1, \dots, w_m \in W_m$.

In particular, 0 can be written uniquely as $w_1 \in W_1, \dots, w_m \in W_m$. Then

$0 \in W_1 + \dots + W_m = 0 \in W_1 + \dots + 0 \in W_m$.

"$\Leftarrow$" Suppose 0 can be written uniquely as

$0 = 0 + \dots + 0$.

Suppose $w_1 + \dots + w_m = w_1' + \dots + w_m'$ as a base.

Subtracting the RHS. from LHS. we obtain

$w_1 + \dots + w_m - (w_1' + \dots + w_m') = 0$

$\Rightarrow (w_1 - w_1') + \dots + (w_m - w_m') = 0$

Now $w_1 - w_1' \in W_1, \dots, w_m - w_m' \in W_m$ since $w_j, w_j' \in W_j \quad \forall 1 \leq j \leq m$.

Since 0 may be written as $0 = 0 + \dots + 0$,

$w_1 - w_1' = 0, \dots, w_m - w_m' = 0$.

$\Rightarrow w_1 = w_1', \dots, w_m = w_m'$. $\square$

Thm. 1.46

Suppose W_1 and W_2 are subspaces of V.

Then $W_1 + W_2 = W_1 \oplus W_2 \iff W_1 \cap W_2 = \{0\}$.

Proof:

"$\Rightarrow$" Suppose $W_1 + W_2 = W_1 \oplus W_2$. Suppose $u \in W_1 \cap W_2$.

Note that $0 + 0 = 0 = u + (-1)u$, $u \in W_1 \cap W_2 \subseteq W_1$ and $(-1)u \in W_1 \cap W_2 \subseteq W_2$.

By thm. 1.45, 0 may be written uniquely as $0+0$ in $W_1 + W_2$ so $u = 0 = (-1)u$.

Thus $W_1 \cap W_2 = \{0\}$.

"$\Leftarrow$" Suppose $W_1 \cap W_2 = \{0\}$.

Using thm. 1.45, we want to show that if $0 = w_1 + w_2$ for $w_1 \in W_1$, $w_2 \in W_2$ then $w_1 = 0$ and $w_2 = 0$.

Suppose $0 = w_1 + w_2$ as above.

Subtracting w_2 from both sides, we obtain

$$0 - w_2 = w_1 + w_2 - w_2$$

$\Rightarrow -w_2 = w_1$.

Now $w_1 \in W_1$ and $-w_2 \in W_2$ and they are equal.

Therefore $w_1 \in W_1$ and $w_2 \in W_2$, i.e.

$$w_1 \in W_1 \cap W_2 = \{0\}$$

$\Rightarrow w_1 = 0$.

Similarily $w_2 \in W_2$ and $w_2 = -w_1 \in W_1$

$\Rightarrow w_2 \in W_1 \cap W_2 = \{0\}$

$\Rightarrow w_2 = 0$. $\square$

Note:

① It is not true that $W_1 + \dots + W_m = W_1 \oplus \dots \oplus W_m \iff W_j \cap W_k = \{0\}$ $\forall 1 \leq j \neq k \leq m$ for $m \geq 3$.

② Later we will define an external direct sum of vector spaces, it uses the same notation but a different definition.

Def. 2.4

Suppose $v_1, \dots, v_m \in V$. Then

$$\text{span}\{v_1, \dots, v_m\} = \{a_1 v_1 + \dots + a_m v_m : a_1, \dots, a_m \in \mathbb{F}\}.$$

We set $\text{span}\,\emptyset = \{0\}$.

Thm. 2.6

Suppose $v_1, \dots, v_m \in V$. Then

$\text{span}\{v_1, \dots, v_m\}$ is a subspace of V and it is the smallest subspace containing $\{v_1, \dots, v_m\}$.

Sketch of proof: subspace test.

Ex.

Set $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \in \mathbb{F}^n$. Then $\text{span}\{e_1, \dots, e_n\} = \mathbb{F}^n$.

$\{e_1, \dots, e_n\}$ is called the standard basis of $\mathbb{F}^n$.

Exercise:

Let $P_m(\mathbb{F})$ = the set of polynomials with coefficients in $\mathbb{F}$ of degree m or less.

$:= \{a_0 + a_1 x + \dots + a_m x^m : a_0, \dots, a_m \in \mathbb{F}\}$.

$P_m(\mathbb{F})$ is a vector space.

Clearly, $P_m(\mathbb{F}) = \text{span}\{1, x, x^2, \dots, x^m\}$.

Let $P(\mathbb{F})$ be the set of all polynomials with coefficients in $\mathbb{F}$, ie. $P(\mathbb{F}) = \bigcup_{m=0}^{\infty} P_m(\mathbb{F})$.

(Continued in next lecture.)