<u>Ex.</u> continued from last lecture.

$\mathbb{P}(\mathbb{F}) = \{a_0 + a_1 x + \dots + a_n x^n : a_0, \dots, a_n \in \mathbb{F}, n \geq 0\}$

We will show that $\mathbb{P}(\mathbb{F})$ is not spanned by a finite number of polynomials.

By way of contradiction, suppose

$$P(\mathbb{F}) = \text{span}\{f_1(x), \dots, f_m(x)\}$$

for some $f_1(x), \dots, f_m(x) \in P(\mathbb{F})$.

Let $d_j = \deg f_j(x)$, $1 \leq j \leq m$.

Recall that $\deg(f(x) + g(x)) \leq \max\{\deg(f(x)), \deg(g(x))\}$ and $\deg(c f(x)) \leq \deg(f(x))$.

It follows that

$$\deg(a_1 f_1(x) + \dots + a_m f_m(x)) \leq \max\{d_1, \dots, d_m\}$$

for any $a_1, \dots, a_m \in \mathbb{F}$.

This implies that
$$P(\mathbb{F}) = \text{span}\{f_1(x), \dots, f_m(x)\} \subseteq P_d(\mathbb{F})$$

where $d = \max\{d_1, \dots, d_m\}$.

This is a contradiction.

<u>Def.</u> 2.15

$\{v_1, \dots, v_m\}$ is <u>linearly independent</u> if

$$a_1 v_1 + \dots + a_m v_m = 0$$
$$\Rightarrow a_1 = \dots = a_m = 0.$$

We define the empty set $\emptyset$ to be linearly independent.

<u>Ex.</u>

$\{e_1, \dots, e_n\}$ is linearly independent in $\mathbb{F}^n$.

$\{1, x, \dots, x^n\}$ is linearly independent in $P_n(\mathbb{F})$.

<u>Exercise</u>

If $\{v_1, \dots, v_m\}$ is lin. ind. then so is every one of its subsets.

<u>Def.</u> 2.17

$\{v_1, \dots, v_n\} \subseteq V$ is linearly dependent if it is not linearly independent.

i.e. $\exists a_1, \dots, a_n \in F$ where not all are zero such that $a_1 v_1 + \dots + a_m v_m = 0$.

<u>Ex.</u>

$\{0\}$ is linearly dependent since $a 0 = 0$ $\forall a \in \mathbb{F}$. In particular, $1 \cdot 0 = 0$.

More generally, $\{0, v_1, \dots, v_m\}$ is lin. dep. since $1 \cdot 0 + 0 v_1 + \dots + 0 v_m = 0$.

Lemma 2.19 (Modified)

Suppose $\{v_1, \dots, v_m\} \subseteq V$. Then

$\{v_1, \dots, v_m\}$ is lin. dep. $\Leftrightarrow$ $\exists\, 1 \leq k \leq m$ s.t. $v_k \in \operatorname{span}\{v_1, \dots, v_{k-1}\}$.

Furthermore, in this case

$\operatorname{span}\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_m\} = \operatorname{span}\{v_1, \dots, v_m\}$.

Proof omitted.

The contrapositive statement:

$\{v_1, \dots, v_m\}$ is lin. ind. $\Leftrightarrow$ none of the vectors is in the span of the earlier ones.

Corollary (2.30)

Suppose $V = \operatorname{span}\{v_1, \dots, v_m\}$. Then

$\{v_1, \dots, v_m\}$ contains a subset $B \subseteq \{v_1, \dots, v_m\}$ with $\operatorname{span} B = V$ and B is linearly independent.

Sketch of proof:

By induction on m.

Induction step:

If $\{v_1, \dots, v_m\}$ is lin. ind., take $B = \{v_1, \dots, v_m\}$ and we are done.

Otherwise $\{v_1, \dots, v_m\}$ is lin. dep. and by lemma 2.19 we can remove a v_k and get

$\operatorname{span}\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_m\} = V$.

The set $\{v_1, \dots, v_{k-1}, v_{k+1}, \dots, v_m\}$ has $m-1$ elements so we may use the induction assumption.

Therefore there is a lin. ind. subset

$B \subseteq \{v_1, \dots, v_{k-1}, v_k, \dots, v_m\}$

with $\operatorname{span} B = V$.

Thm. 2.22 (Modified)

Suppose $\operatorname{span}\{v_1, \dots, v_m\} = V$.

If $\{w_1, \dots, w_k\} \subseteq V$ is lin. ind. then $k \leq m$.

(Linearly independent sets are not bigger than spanning sets)

Proof omitted.

The contrapositive statement:

Suppose $\operatorname{span}\{v_1, \dots, v_m\} = V$. If $\{w_1, \dots, w_k\} \subseteq V$ with $k > m$ then $\{w_1, \dots, w_m\}$ is lin. dep.

Ex.

$\operatorname{span}\{e_1, \dots, e_n\} = \mathbb{F}^n$

$\operatorname{span}\{1, x, \dots, x^m\} = \mathcal{P}_m(\mathbb{F})$

Def.

A <u>basis</u> for V is a subset $B \subseteq V$ such that
(i) B is lin. ind.
(ii) $\operatorname{span} B = V$.

The <u>dimension</u> $\dim V$ is the size $|B|$ of B for any basis B of V.

V is <u>finite-dimensional</u> if V has a finite basis.

V is <u>infinite-dimensional</u> if V does not have a finite basis.

Ex.

- $\dim \mathbb{F}^n = n$ since the basis $\{e_1, \dots, e_n\}$ has n elements.
- $\dim P_m(\mathbb{F}) = m+1$ since the basis $\{1, x, \dots, x^m\}$ has $m+1$ elements.
- $\dim P(\mathbb{F}) = \infty$ since it has no finite spanning sets, and therefore no finite bases.

Note

Suppose B and B' are two finite bases for V. Then
$\operatorname{span} B = V$ and B' lin ind $\Rightarrow |B'| \leq |B|$ (thm. 2.22)
Similarily,
$\operatorname{span} B' = V$ and B lin ind. $\Rightarrow |B| \leq |B'|$.
Therefore $|B| = |B'|$ and dimension is well-defined.

Thm 2.32 (Modified)

Suppose $\dim V = n < \infty$ and $W \subseteq V$ is a subspace.
Then ① $\dim W \leq \dim V < \infty$
② Every basis for W extends to a basis for V.
③ Every lin. ind subset of V extends to a basis for V.

Sketch of proof:
Proof of ①:
Suppose $W = \{0\}$. Then $\dim W = 0 < \infty$.
Otherwise suppose $W \neq \{0\}$.
Pick $w_1 \in W$, $w_1 \neq 0$.
Then $\{w_1\}$ is lin. ind.
If $\operatorname{span}\{w_1\} = W$ then $\{w_1\}$ is a basis and $\dim W = 1 < \infty$.
Otherwise $\exists w_2 \notin \operatorname{span}\{w_1\}$, $w_2 \in W$.
By lemma 2.19 $\{w_1, w_2\}$ is lin. ind.
If $\operatorname{span}\{w_1, w_2\} = W$ then $\dim W = 2 < \infty$.
We can repeat this process at most $\dim V$ times, since theorem 2.22 puts an upper limit to the size of $\{w_1, \dots, w_k\}$ to be $\dim V$.

In other words,

$\operatorname{span}\{w_1, \dots, w_k\} = W$ for some $k \leq \dim V$.

In particular $\dim W = |\{w_1, \dots, w_k\}| = k \leq \dim V$.

Proof of ②:

Continue the process in ①, but now asking where the span is equal to V.

For example if $\operatorname{span}\{w_1, \dots, w_k\} = V$ then we are done since $\{w_1, \dots, w_k\}$ is a basis for V.

Otherwise $\operatorname{span}\{w_1, \dots, w_k\} \neq V$ so $\exists v_1 \in V$, $v_1 \notin \operatorname{span}\{w_1, \dots, w_k\}$ such that

$\{w_1, \dots, w_k, v_1\}$ is lin. ind.

Continue the process with $\{w_1, \dots, w_k, v_1\}$. The process must terminate by thm. 2.22.

Proof of ③:

Suppose $\{u_1, \dots, u_\ell\} \subseteq V$ is lin. ind. Then set $W = \operatorname{span}\{u_1, \dots, u_\ell\}$ and apply ②.

Thm. 2.33

Suppose $\dim V = n < \infty$ and $W \subseteq V$ is a subspace.

Then there exists a subspace $U \subseteq V$ such that

$V = W \oplus U$.

Proof: next lecture.