Def.

$A \in M_{nn}(F)$ is diagonizable if
$$A = SDS^{-1}$$
for some diagonal matrix $D \in M_{nn}(F)$

Thm. DC

$A \in M_{nn}(F)$ is diagonizable $\Leftrightarrow$ there is a basis $\{v_1, \ldots, v_n\}$ for F^n consisting of eigenvectors of A.
$(Av_j = \lambda_j v_j \ \forall 1 \le j \le n)$

Proof

"$\Rightarrow$" $A = SDS^{-1} \Leftrightarrow AS = SD$

Set $S = [v_1 \cdots v_n]$, $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$. Then
$$AS = [Av_1 \cdots Av_n], \text{ and}$$
$$SD = \ldots = [\lambda_1 v_1 \cdots \lambda_n v_n]$$
So $AS = SD \Rightarrow Av_j = \lambda_j v_j \ \forall 1 \le j \le n$

"$\Leftarrow$" Suppose $\{v_1, \ldots, v_n\}$ is a basis for F^n consisting of eigenvectors of A. Suppose $Av_j = \lambda_j v_j \ \forall 1 \le j \le n$ with $\lambda_1, \ldots, \lambda_n \in F$.

Let $S = [v_1 \cdots v_n] \in M_{nn}(F)$. By thm NME, S is invertible as $\{v_1, \ldots, v_n\}$ is a basis.

Let $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$.

As before,
$$AS = [Av_1 \cdots Av_n]$$
$$= [\lambda_1 v_1 \cdots \lambda_n v_n]$$
$$= SD$$
And $AS = SD \Leftrightarrow ASS^{-1} = SDS^{-1} \Leftrightarrow A = SDS^{-1}$.

Remark:

The proof above tells us that if $A = SDS^{-1}$ with D diagonal then the columns of S are a basis of eigenvectors and the diagonal entries of D are the respective eigenvalues.

Ex. DMS 3

Take $A = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \in M_{33}(\mathbb{C})$.

Then $p_A(x) = \ldots = -(x-3)(x+1)^2$ (example EMS 3)

$\Rightarrow$ eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -1$

Then find eigenspaces to find eigenvectors.

$\mathcal{E}_A(3) = \mathcal{N}(A - 3I) = \text{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}\right\}$ (Take for granted these vectors form a basis together.

$\mathcal{E}_A(-1) = \mathcal{N}(A + 1I) = \text{span}\left\{\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}\right\}$. Proof of this in a future theorem)

Then take $S = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 3 & 0 \\ 2 & 0 & 3 \end{bmatrix}$, S^{-1} exists by thm. NME. Take $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

Thus by theorem DC, $A = SDS^{-1}$.

Thm. DED

Suppose $A \in M_{nn}(F)$ has n distinct eigenvalues $\lambda_1, \dots, \lambda_n \in F$.

Then A is diagonizable.

Proof:

Let $v_1, \dots, v_n$ be eigenvectors corresponding to $\lambda_1, \dots, \lambda_n$ respectively.

ie. $Av_j = \lambda_j v_j \ \forall 1 \le j \le n$.

By thm. EDELI, $\{v_1, \dots, v_n\}$ is lin. ind.

By thm. G, $\{v_1, \dots, v_n\}$ is a basis of eigenvectors.

By thm. DC, A is diagonalizable.

Ex. Take $A = \begin{bmatrix} 2 & 500.2 \\ 0 & -4 \end{bmatrix} \in M_{22}(\mathbb{R})$.

Then A has 2 and -4 as eigenvalues, so by thm DED, A is diagonalizable.

Ex. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ has only 1 as an eigenvalue, however $I = III^{-1}$, so I is diagonalizable.

Ex. D diagonal $\in M_{nn}(F)$ is diagonalizable since $D = IDI^{-1}$.

Thm. (not in textbook)

Suppose A has k distinct eigenvalues $\lambda_1, \dots, \lambda_k \in F$, and let $B_1, \dots, B_k$ be bases for $\mathcal{E}_A(\lambda_1), \dots, \mathcal{E}_A(\lambda_k)$ respectively.

Then the following are equivalent.

(i) A is diagonalizable

(ii) $B_1 \cup \dots \cup B_k$ is a basis for F^n.

(iii) $n = \gamma_A(\lambda_1) + \dots + \gamma_A(\lambda_k) = \dim \mathcal{E}_A(\lambda_1) + \dots + \dim \mathcal{E}_A(\lambda_k)$

(iv) $p_A(x) = (-1)^n (x-\lambda_1)^{\gamma_A(\lambda_1)} \cdots (x-\lambda_k)^{\gamma_A(\lambda_k)}$

(v) $\gamma_A(\lambda_j) = \alpha_A(\lambda_j) \ \forall 1 \le j \le k$ and $\sum_{j=1}^{k} \alpha_A(\lambda_j) = n$.

Proof: For simplicity, take $k=2$

"(i) $\Rightarrow$ (v)" Suppose $A = SDS^{-1}$. By thm. DC, there is a basis for F^n consisting of eigenvectors.

Let $v_1, \dots, v_\ell$ be the eigenvectors in this basis with $Av_j = \lambda_1 v_j$ for $1 \le j \le \ell$. $(\ell + m = n)$

Let $w_1, \dots, w_m$ be the eigenvectors in this basis with $Aw_j = \lambda_2 w_j$ for $1 \le j \le m$.

Then $A = SDS^{-1}$ where $S = [v_1 \cdots v_\ell \, w_1 \cdots w_m]$ and $D = \begin{bmatrix} \lambda_1 \ddots \lambda_1 & 0 \\ 0 & \lambda_2 \ddots \lambda_2 \end{bmatrix}$.

By thm. SMEE, $p_A(x) = \operatorname{Det}(D - xI) = (\lambda_1 - x)^\ell (\lambda_2 - x)^m$.

$(\lambda_2 - x)^m$ does not have λ_1 as a root, $(\lambda_1 - x)$ does not divide $(\lambda_2 - x)^m$. Therefore

ℓ = highest power of $(\lambda_1 - x)$ in $p_A(x) = \alpha_A(\lambda_1)$.

Similarily, $m = \alpha_A(\lambda_2)$

Note that $\{v_1, \dots, v_\ell\}$ is lin. ind. as it is contained in the basis $\{v_1, \dots, v_\ell, w_1, \dots, w_m\}$

Since $\{v_1, \dots, v_\ell\} \subseteq \mathcal{E}_A(\lambda_1)$,

$$\ell \le \dim \mathcal{E}_A(\lambda_1) = \underbrace{\gamma_A(\lambda_1) \le \alpha_A(\lambda_1)}_{\text{(thm. ME)}} = \ell \Rightarrow \gamma_A(\lambda_1) = \alpha_A(\lambda_1)$$

Similarily $m \le \dim \mathcal{E}_A(\lambda_2) = \gamma_A(\lambda_2) \le \alpha_A(\lambda_2) = m \Rightarrow \gamma_A(\lambda_2) = \alpha_A(\lambda_1)$.

Proof continued

"(v) $\Rightarrow$ (iv)" Suppose (v).

By definition of $\alpha_A(\lambda_j)$, $p_A(x) = (\lambda_1 - x)^{\alpha_A(\lambda_1)} (\lambda_2 - x)^{\alpha_A(\lambda_2)} q(x)$

where $q(\lambda_1) \neq 0$ and $q(\lambda_2) \neq 0$. By (v), $\alpha_A(\lambda_1) + \alpha_A(\lambda_2) = n = \deg p_A(x)$

$\Rightarrow \deg q(x) = 0 \Rightarrow q(x)$ is constant.

$\Rightarrow q(x) = 1$ (since leading term is $(-1)^{\alpha_A(\lambda_1)+\alpha_A(\lambda_2)}$).

$$\begin{aligned} \Rightarrow p_A(x) &= (\lambda_1 - x)^{\alpha_A(\lambda_1)} (\lambda_2 - x)^{\alpha_A(\lambda_2)} \\ &= (\lambda_1 - x)^{\gamma_A(\lambda_1)} (\lambda_2 - x)^{\gamma_A(\lambda_2)} \qquad \text{(by (v))} \\ &= (-1)^n (x-\lambda_1)^{\gamma_A(\lambda_1)} (x - \lambda_2)^{\gamma_A(\lambda_2)} \end{aligned}$$