Thm. end of last lecture.

Proof for $k=2$:

"(i) $\Rightarrow$ (v)": done

"(v) $\Rightarrow$ (iv)": done

"(iv) $\Rightarrow$ (iii)":

By (iv), $\deg(p_A(x)) = \deg\left((\lambda_1 - x)^{\gamma_A(\lambda_1)} \cdots (\lambda_k - x)^{\gamma_A(\lambda_k)}\right)$

$$\Rightarrow \quad n = \deg\left((\lambda_1 - x)^{\gamma_A(\lambda_1)}\right) + \dots + \deg\left((\lambda_k - x)^{\gamma_A(\lambda_k)}\right)$$
$$= \gamma_A(\lambda_1) + \dots + \gamma_A(\lambda_k).$$

"(iii) $\Rightarrow$ (ii)": Take $k=2$

Suppose $n = \gamma_A(\lambda_1) + \gamma_A(\lambda_2)$

Recall $\gamma_A(\lambda_1) = \dim \mathcal{E}_\lambda(\lambda_1) = |B_1| = \#$ vectors in B_1

Similarly, $\gamma_A(\lambda_2) = \dim \mathcal{E}_A(\lambda_2) = |B_2| = \#$ vectors in B_2.

$B_1 \cap B_2 = \emptyset$ since you can't have an eigenvector with 2 different eigenvalues.

$|B_1 \cup B_2| = |B_1| + |B_2| = \gamma_A(\lambda_1) + \gamma_A(\lambda_2) = n = \dim F^n$.

In order to show that $B_1 \cup B_2$ is a basis for F^n, it suffices to check that $B_1 \cup B_2$ is linearly independent (thm. G) (Assignment 5)

"(ii) $\Rightarrow$ (i)":

Suppose $B_1 \cup B_2$ is a basis for F^n with $B_1 = \{v_1, \dots, v_\ell, w_1, \dots, w_m\}$, $\ell + m = n$.

Let $S = [v_1 \cdots v_\ell \; w_1 \cdots w_m]$.

Following the proof of thm. DC, we obtain $A = SDS^{-1}$, with $S = \begin{bmatrix} \lambda_1 & & & & 0 \\ & \ddots \lambda_1 & & & \\ & & \lambda_2 & & \\ 0 & & & \ddots & \lambda_2 \end{bmatrix}$

Vector representation

Idea For an n-dimensional vector space V with $B = \{v_1, \dots, v_n\}$ as a basis for V, we would like to convert information about V into information about F^n (and viceversa) using an isomorphism $P_B : V \to F^n$.

Def. (From 1152)

An isomorphism between two vector spaces V and W (over F) is a linear transformation $T: V \to W$ which is both injective and surjective. In this case, we write $V \cong W$ and

Thm. ILTIS

Suppose $T: U \to W$ is a linear transfer

Then T is an isomorphism $\Leftrightarrow$ T is invertible ($\exists T^{-1}: W \to V$ such that $T \circ T^{-1}(w) = w$ $\forall w \in W$, and $T^{-1} \circ T(v) = v$).

Thm. ILTLT

$T \in \mathcal{LT}(V, W)$ is invertible $\Rightarrow$ $T^{-1} \in \mathcal{LT}(W, V)$ (T^{-1} is also linear).

Idea: If $V \cong W$, then the linear algebra of U is identical to the linear algebra of W.

Def. VR

Define ρ_B as follows:

Let $w \in V$. Then $w = \alpha_1 v_1 + \dots + \alpha_n v_n$ for unique choice of $\alpha_1, \dots, \alpha_n \in F$.

Define $\rho_B(w) = \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix} \in F^n$.

In other words, $w = [\rho_B(w)]_1 v_1 + \dots + [\rho_B(w)]_n v_n = \sum_{\ell=1}^{n} [\rho_B(w)]_\ell v_\ell$

ρ_B is called the coordinate map (relative to B)

Ex. Let $P_2 = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in F\}$

Let $B = \{1, 1+x, x^2\}$

Then $a_0 + a_1 x + a_2 x^2 = (a_0 - a_1) + a_1(1+x) + a_2 x^2$

$\Rightarrow \rho_B(a_0 + a_1 x + a_2 x^2) = \begin{bmatrix} a_0 - a_1 \\ a_1 \\ a_2 \end{bmatrix}$.

Ex. Take $V = M_{22}(F)$ with $B = \left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$

$\rho_B\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right) = \begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ since

$\begin{bmatrix}a&b\\c&d\end{bmatrix} = a\begin{bmatrix}1&0\\0&0\end{bmatrix} + b\begin{bmatrix}0&1\\0&0\end{bmatrix} + c\begin{bmatrix}0&0\\1&0\end{bmatrix} + d\begin{bmatrix}0&0\\0&1\end{bmatrix}$

Ex. VRC4

Take $V = \mathbb{C}^4$, $B = \left\{ \overset{v_1}{\begin{bmatrix}-2\\1\\2\\3\end{bmatrix}}, \overset{v_2}{\begin{bmatrix}3\\-6\\2\\4\end{bmatrix}}, \overset{v_3}{\begin{bmatrix}1\\2\\0\\5\end{bmatrix}}, \overset{v_4}{\begin{bmatrix}4\\3\\1\\6\end{bmatrix}} \right\}$, a basis of $\mathbb{C}^4$.

Take $w = \begin{bmatrix}6\\14\\6\\7\end{bmatrix} \in \mathbb{C}^4$

To find $\rho_B(w)$, need to solve $c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = w$.

$\Rightarrow$ RREF of $[v_1\ v_2\ v_3\ v_4 \mid w]$

$\Rightarrow \begin{bmatrix}c_1\\c_2\\c_3\\c_4\end{bmatrix} = [$

Thm. VRLT

$\rho_B : V \to F^n$ is a linear transformation

Proof. Suppose $u, w \in V$

By def, $u = \sum_{\ell=1}^{n} [\rho_B(u)]_\ell v_\ell$, $w = \sum_{\ell=1}^{n} [\rho_B(w)]_\ell v_\ell$

$\Rightarrow u + w = \sum_{\ell=1}^{n} [\rho_B(u)]_\ell v_\ell + [\rho_B(w)]_\ell v_\ell$

$= \sum_{\ell=1}^{n} ([\rho_B(u)]_\ell + [\rho_B(w)]_\ell) v_\ell$

$\rho_B(u+w) = \begin{bmatrix}[\rho_B(u)]_1 + [\rho_B(w)]_1 \\ \vdots \\ [\rho_B(u)]_n + [\rho_B(w)]_n\end{bmatrix} = \begin{bmatrix}[\rho_B(u)]_1 \\ \vdots \\ [\rho_B(u)]_n\end{bmatrix} + \begin{bmatrix}[\rho_B(w)]_1 \\ \vdots \\ [\rho_B(w)]_n\end{bmatrix} = \rho_B(u) + \rho_B(w)$.

Exercise: prove $\rho_B(\alpha w) = \alpha \rho_B(w)$ $\forall \alpha \in F$. (Assignment 5).

Recall: $T \in LT(V, W)$ is injective $\Leftrightarrow$ $\ker T = \{v \in V : T(v) = 0\}$ is $\{0\}$ (thm. KILT).

Thm. VRI

$\rho_B \in LT(V, F^n)$ is injective.

Proof.

$\ker \rho_B = \left\{v \in V : \rho_B(v) = \begin{bmatrix}0\\ \vdots \\ 0\end{bmatrix}\right\}$

By definition $\rho_B(v) = 0 \Leftrightarrow v = 0v_1 + 0v_2 + \dots + 0v_n \Leftrightarrow v = 0$.

Thus $\ker \rho_B = \{0\}$ and ρ_B is injective by thm. KILT.

Thm VRS

$\rho_B \in LT(V, F^n)$ is surjective.

Proof:

Suppose $y \in F^n$ where $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in F^n$.

Let $v = y_1 v_1 + y_2 v_2 + \ldots + y_n v_n \in V$

By construction, $\rho_B(v) = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$

Corollary VRILT

$\rho_B \in LT(V, F^n)$ is an isomorphism.

In particular, $V \cong F^n$.

Proof: Thm. VRS, VRI, VRLT.