Def.

For a vector space V over F with basis $B = \{v_1, \dots, v_n\}$.
The coordinate map $\rho_B : V \to F^n$ is defined by
$w = \sum_{\ell=1}^{n} [\rho_B(w)]_\ell v_\ell$

Corollary VRILT
$\rho_B \in LT(V, F^n)$ is an isomorphism.
In particular, $V \cong F^n$.

Thm. IFDVS
Suppose W and V are vector spaces with $\dim W = \dim V = n$.
Then $V \cong W$.
Proof
Fix a basis $C = \{w_1, \dots, w_n\}$ for W.
By corollary VRILT, we have two isomorphisms $\rho_B : V \to F^n$ and $\rho_C : W \to F^n$.
Since ρ_C is invertible (thm. ILTLT) and $\rho_C^{-1} : F^n \to W$ is an isomorphism, we can compose them to obtain an isomorphism (thm. CIVLT)
$\rho_C^{-1} \circ \rho_B : V \to W$
By def, now $V \cong W$.

Thm IVSED
Suppose V and W are finite dimensional vector spaces and $V \cong W$.
Then $\dim V = \dim W$.
Proof
Let $T \in LT(V, W)$ be an isomorphism.
Let $B = \{v_1, \dots, v_n\}$ be a basis for V.
Let's prove that $\{T(v_1), \dots, T(v_n)\}$ is a basis for W.
To show lin. ind. suppose
$\alpha_1 T(v_1) + \dots + \alpha_n T(v_n) = 0$
$\Rightarrow T(\alpha_1 v_1 + \dots + \alpha_n v_n) = 0$ (T is linear)
$\Rightarrow \alpha_1 v_1 + \dots + \alpha_n v_n + \ker T = \{0\}$ (T is injective, thm. KILT)
$\Rightarrow \alpha_1 v_1 + \dots + \alpha_n v_n = 0$
$\Rightarrow \alpha_1 = \dots = \alpha_n = 0$ (Since $\{v_1, \dots, v_n\}$ is lin. ind.)
To prove that $\operatorname{span}\{T(v_1), \dots, T(v_n)\} = W$, suppose $w \in W$.
Since T is surjective, there exists $v \in V$ such that $T(v) = w$.
Since $\operatorname{span}\{v_1, \dots, v_n\} = V$, $\exists\, \beta_1, \dots, \beta_n \in F$ such that
$v = \beta_1 u_1 + \dots + \beta_n u_n$
$\Rightarrow T(v) = T(\beta_1 u_1 + \dots + \beta_n u_n)$
$\Rightarrow w = \beta_1 T(u_1) + \dots + \beta_n T(u_n) \in \operatorname{span}\{T(v_1), \dots, T(v_n)\}$
Thus $\operatorname{span}\{T(v_1), \dots, T(v_n)\} = W$ and is a basis for W. Since it has n vectors, $\dim W = n = \dim V$.

Remarks

- For linear algebra of finite dimensional vector spaces, all that matters is F and the dimension.

Thm. CLI, CSS

Suppose $u_1, \dots, u_k \in V$ and B is a basis for V.

(a) $\{u_1, \dots, u_k\}$ is lin. ind. $\Leftrightarrow$ $\{p_B(u_1), \dots, p_B(u_k)\}$ is lin. ind.

(b) $\operatorname{span}\{u_1, \dots, u_k\} = V$ $\Leftrightarrow$ $\operatorname{span}\{p_B(u_1), \dots, p_B(u_k)\} = F^n$.

Proof

Take $W = F^n$ and $T = p_B$ in the proof of the previous theorem.

Matrix Representations

Idea: We can easily convert n-dimensional vector space V into F^n using coordinate map p_B.

Suppose $T \in \mathcal{L}T(V, W)$ is a lin. trans. on finite dimensional vector spaces.

Can we convert T into a linear transformation between F^n and F^m? We should get a matrix.

Ex. Suppose V is a 2-dimensional vector space with basis $B = \{v_1, v_2\}$.

Let $T \in \mathcal{L}T(V, V)$. Suppose $v \in V$.

Recall that $v = [p_B(v)]_1 v_1 + [p_B(v)]_2 v_2$

$$\Rightarrow T(v) = T([p_B(v)]_1 v_1 + [p_B(v)]_2 v_2)$$
$$= [p_B(v)]_1 T(v_1) + [p_B(v)]_2 T(v_2)$$

$$\Rightarrow p_B(T(v)) = p_B([p_B(v)]_1 T(v_1) + [p_B(v)]_2 T(v_2))$$
$$= [p_B(v)]_1 p_B(T(v_1)) + [p_B(v)]_2 p_B(T(v_2))$$
$$= [p_B(T(v_1)) \;\; p_B(T(v_2))]\begin{bmatrix} [p_B(v)]_1 \\ [p_B(v)]_2 \end{bmatrix}$$
$$= [p_B(T(v_1)) \;\; p_B(T(v_2))]\, p_B(v)$$

$$\begin{tikzcd} v \arrow[r, mapsto, "T"] \arrow[d, "p_B"'] & T(v) \arrow[d, "p_B"] \\ p_B(v) \arrow[r, dotted] & p_B(T(v)) \end{tikzcd}$$

multiplication by $[p_B(T(v_1)) \;\; p_B(T(v_2))] \in M_{22}(F)$.

Ex. Take $V = P_1 = \{a_0 + a_1 x : a_0, a_1 \in \mathbb{R}\}$

Define $T: P_1 \to P_1$ as the derivative. $T(a_0 + a_1 x) = a_1$.

Let $B = \{1, x\}$. Take $v_1 = 1$, $v_2 = x$.

We compute $[p_B(T(v_1)) \;\; p_B(T(v_2))] = [p_B(T(1)) \;\; p_B(T(x))]$
$$= [p_B(0) \;\; p_B(1)]$$

$0 = 0 + 0x \Rightarrow p_B(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$1 = 1 + 0x \Rightarrow p_B(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Then we get $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \in M_{22}(\mathbb{R})$.

$p_B(a_0 + a_1 x) = \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$.

$\Rightarrow [p_B(1) \;\; p_B(x)]\, p_B(a_0 + a_1 x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \end{bmatrix}$.

Def. MR

Suppose V has basis $B = \{v_1, \dots, v_n\}$, W has basis $\mathcal{C} = \{w_1, \dots, w_m\}$ and $T \in \mathcal{L}T(V, W)$.

Then the matrix representation of T (relative to B and $\mathcal{C}$) is

$$[T]_B^{\mathcal{C}} = [p_{\mathcal{C}}(T(v_1)) \;\cdots\; p_{\mathcal{C}}(T(v_n))] \in M_{mn}(F).$$