

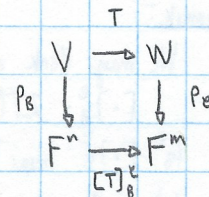
Recall: V is a vector space over F . $B = \{v_1, \dots, v_n\}$ is a basis for V . $p_B: V \rightarrow F^n$ is defined by

$$w = [p_B(w)]_1 v_1 + [p_B(w)]_2 v_2 + \dots + [p_B(w)]_n v_n, \quad w \in V.$$

$$= \sum_{j=1}^n [p_B(w)]_j v_j, \quad p_B(w) = \begin{bmatrix} [p_B(w)]_1 \\ \vdots \\ [p_B(w)]_n \end{bmatrix}$$

 $p_B: V \rightarrow F^n$ and $p_B^{-1}: F^n \rightarrow V$ are both isomorphisms.Def. MRSuppose $T \in \mathcal{L}(V, W)$, i.e. $T: V \rightarrow W$ is linear.Suppose $B = \{v_1, \dots, v_n\}$ and $C = \{w_1, \dots, w_m\}$ are bases for V and W respectively.Then the matrix representation of T with respect to B and C is

$$[T]_B^C = [p_C(T(v_1)) \dots p_C(T(v_n))] \in M_{m,n}(F)$$

Ex Take $F = \mathbb{R}$, $V = \mathbb{R}^3$, $W = P_1$ = polynomials of degree 1 or less.Let $B = \{e_1, e_2, e_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $C = \{1, x\}$ Define $T: \mathbb{R}^3 \rightarrow P_1$ by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (2a-b) + (a+b+c)x \quad \forall \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$$

Take for granted that T is linear.Let's compute $[T]_B^C$:

$$[p_C(T(e_1)) \ p_C(T(e_2)) \ p_C(T(e_3))] = [p_C(2(1) + (1)x) \ p_C(-1 + x) \ p_C(0 + x)]$$

$$= \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \in M_{2,3}(\mathbb{R})$$

Thm FTMRSuppose $T \in \mathcal{L}(V, W)$, $B = \{v_1, \dots, v_n\}$ is a basis for V , $C = \{w_1, \dots, w_m\}$ is a basis for W .Then $p_C(T(v)) = [T]_B^C p_B(v) \quad \forall v \in V$.

(i.e. the diagram from above commutes).

Proof:Let $v \in V$. Then $v = \sum_{j=1}^n [p_B(v)]_j v_j$

$$\Rightarrow T(v) = T\left(\sum_{j=1}^n [p_B(v)]_j v_j\right)$$

$$= \sum_{j=1}^n [p_B(v)]_j T(v_j)$$

$$\Rightarrow p_C(T(v)) = p_C\left(\sum_{j=1}^n [p_B(v)]_j T(v_j)\right)$$

$$= \sum_{j=1}^n [p_B(v)]_j p_C(T(v_j))$$

$$= [p_C(T(v_1)) \dots p_C(T(v_n))] \begin{bmatrix} [p_B(v)]_1 \\ \vdots \\ [p_B(v)]_n \end{bmatrix}$$

$$= [T]_B^C p_B(v) \quad \square$$

Ex. Continuing from the previous example, we compute

$$[T]_B^B p_B \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a-b \\ a+b+c \end{bmatrix}$$

On the other hand,

$$p_B(T(\begin{bmatrix} a \\ b \\ c \end{bmatrix})) = p_B((2a-b) + (a+b+c)x) = \begin{bmatrix} 2a-b \\ a+b+c \end{bmatrix}$$

Remark:

Every property of a linear transformation $T \in \mathcal{LT}(V, W)$ should be connected to an equivalent property for the matrix $[T]_B^B \in M_{mn}(F)$, and vice versa.

Remark:

We may think of the matrix representation of a linear transformation as

$$[\cdot]_B^B: \mathcal{LT}(V, W) \rightarrow M_{mn}(F)$$

Is $[\cdot]_B^B$ linear? (Yes)

We won't prove it, but it is also an isomorphism.

Thm. MRSLT, MRMLT

Suppose $T, S \in \mathcal{LT}(V, W)$, $\alpha \in F$ and $B = \mathcal{C}$ are bases as before. Then

$$[\alpha T + S]_B^B = \alpha [T]_B^B + [S]_B^B$$

Proof.

Let $v \in V$. Then by thm. FTMR,

$$\begin{aligned} [\alpha T + S]_B^B p_B(v) &= p_B((\alpha T + S)(v)) \\ &= p_B(\alpha T(v) + S(v)) \\ &= \alpha p_B(T(v)) + p_B(S(v)) \\ &= \alpha [T]_B^B p_B(v) + [S]_B^B p_B(v) \\ &= (\alpha [T]_B^B + [S]_B^B) p_B(v) \end{aligned}$$

Recall that p_B is surjective so every vector $x \in F^n$ can be written as $p_B(v)$ for some $v \in V$.

We can replace the above with

$$(\alpha [T]_B^B + [S]_B^B)$$