Recall:

V is a vector space over F.

$B=\{v_1,\dots,v_n\}$ is a basis for V.

$p_B: V\to F^n$ is defined by

$$w=[p_B(w)]_1v_1+[p_B(w)]_2v_2+\dots+[p_B(w)]_nv_n,\quad w\in V.$$
$$=\sum_{j=1}^n [p_B(w)]_j v_j,\qquad p_B(w)=\begin{bmatrix}[p_B(w)]_1\\ \vdots\\ [p_B(w)]_n\end{bmatrix}$$

$p_B: V\to F^n$ and $p_B^{-1}: F^n\to V$ are both isomorphisms.

Def. MR

Suppose $T\in\mathcal{L}(V,W)$, i.e. $T:V\to W$ is linear.

Suppose $B=\{v_1,\dots,v_n\}$ and $\mathcal{C}=\{w_1,\dots,w_m\}$ are basis for V and W respectively.

Then the matrix representation of T with respect to B and $\mathcal{C}$ is

$$[T]_B^{\mathcal{C}}=[p_{\mathcal{C}}(T(v_1))\ \cdots\ p_{\mathcal{C}}(T(v_n))]\in M_{mn}(F)$$

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d, "p_B"'] & W \arrow[d, "p_{\mathcal{C}}"] \\ F^n \arrow[r, "{[T]_B^{\mathcal{C}}}"'] & F^m \end{tikzcd}$$

Ex. Take $F=\mathbb{R}$, $V=\mathbb{R}^3$, $W=P_1$ = polynomials of degree 1 or less.

Let $B=\{e_1,e_2,e_3\}=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ and $\mathcal{C}=\{1,x\}$

Define $T:\mathbb{R}^3\to P_1$ by

$$T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right)=(2a-b)+(a+b+c)x\qquad \forall\begin{bmatrix}a\\b\\c\end{bmatrix}\in\mathbb{R}^3$$

Take for granted that T is linear.

Let's compute $[T]_B^{\mathcal{C}}$:

$$[p_{\mathcal{C}}(T(e_1))\ p_{\mathcal{C}}(T(e_2))\ p_{\mathcal{C}}(T(e_3))]=[p_{\mathcal{C}}(2(1)+(1)x)\ p_{\mathcal{C}}(-1+x)\ p_{\mathcal{C}}(0+x)]$$
$$=\begin{bmatrix}2&-1&0\\1&1&1\end{bmatrix}\in M_{23}(\mathbb{R})$$

Thm FTMR

Suppose $T\in\mathcal{L}(V,W)$, $B=\{v_1,\dots,v_n\}$ is a basis for V, $\mathcal{C}=\{w_1,\dots,w_m\}$ is a basis for W.

Then $p_{\mathcal{C}}(T(v))=[T]_B^{\mathcal{C}}p_B(v)\quad\forall v\in V$.

(i.e. the diagram from above commutes).

Proof:

Let $v\in V$. Then $v=\sum_{j=1}^n[p_B(v)]_jv_j$

$$\Rightarrow T(v)=T\left(\sum_{j=1}^n[p_B(v)]_jv_j\right)$$
$$=\sum_{j=1}^n[p_B(v)]_jT(v_j)$$
$$\Rightarrow p_{\mathcal{C}}(T(v))=p_{\mathcal{C}}\left(\sum_{j=1}^n[p_B(v)]_jT(v_j)\right)$$
$$=\sum_{j=1}^n[p_B(v)]_jp_{\mathcal{C}}(T(v_j))$$
$$=[p_{\mathcal{C}}(T(v_1))\cdots p_{\mathcal{C}}(T(v_n))]\begin{bmatrix}[p_B(v)]_1\\ \vdots\\ [p_B(v)]_n\end{bmatrix}$$
$$=[T]_B^{\mathcal{C}}p_B(v).\qquad\square$$

Ex. Continuing from the previous example, we compute

$$[T]_{\mathcal{B}}^{\mathcal{E}} \, p_{\mathcal{B}}\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a-b \\ a+b+c \end{bmatrix}$$

On the other hand,

$$p_{\mathcal{E}}\left(T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right)\right) = p_{\mathcal{E}}\left((2a-b) + (a+b+c)x\right) = \begin{bmatrix} 2a-b \\ a+b+c \end{bmatrix}$$

Remark:

Every property of a linear transformation $T \in \mathcal{LT}(V,W)$ should be connected to an equivalent property for the matrix $[T]_{\mathcal{B}}^{\mathcal{E}} \in M_{mn}(F)$, and vice versa.

Remark:

We may think of the matrix representation of a linear transformation as

$$[\cdot]_{\mathcal{B}}^{\mathcal{E}} : \mathcal{LT}(V,W) \to M_{mn}(F)$$

Is $[\cdot]_{\mathcal{B}}^{\mathcal{E}}$ linear? (Yes)

We won't prove it, but it is also an isomorphism.

Thm. MRSLT, MRMLT

Suppose $T, S \in \mathcal{LT}(V,W)$, $\alpha \in F$ and $\mathcal{B}$ & $\mathcal{E}$ are bases as before. Then

$$[\alpha T + S]_{\mathcal{B}}^{\mathcal{E}} = \alpha [T]_{\mathcal{B}}^{\mathcal{E}} + [S]_{\mathcal{B}}^{\mathcal{E}}.$$

Proof.

Let $v \in V$. Then by thm. FTMR,

$$\begin{aligned} [\alpha T + S]_{\mathcal{B}}^{\mathcal{E}} \, p_{\mathcal{B}}(v) &= p_{\mathcal{E}}((\alpha T + S)(v)) \\ &= p_{\mathcal{E}}(\alpha T(v) + S(v)) \\ &= \alpha p_{\mathcal{E}}(T(v)) + p_{\mathcal{E}}(S(v)) \\ &= \alpha [T]_{\mathcal{B}}^{\mathcal{E}} \, p_{\mathcal{B}}(v) + [S]_{\mathcal{B}}^{\mathcal{E}} \, p_{\mathcal{B}}(v) \\ &= (\alpha [T]_{\mathcal{B}}^{\mathcal{E}} + [S]_{\mathcal{B}}^{\mathcal{E}}) \, p_{\mathcal{B}}(v) \end{aligned}$$

Recall that $p_{\mathcal{B}}$ is surjective so every vector $x \in F^n$ can be written as $p_{\mathcal{B}}(v)$ for some $v \in V$.

We can replace the above with

$$(\alpha [T]_{\mathcal{B}}^{\mathcal{E}} + [S]_{\mathcal{B}}^{\mathcal{E}}$$