<u>Thm</u> MRLCT

Suppose $T: V \to U$, $S: U \to W$ are linear transformations and D is a basis for U. Then

$$[S \circ T]_B^{\mathcal{C}} = [S]_D^{\mathcal{C}} [T]_B^D$$

<u>Proof</u>:

$$\begin{aligned}[S\circ T]_B^{\mathcal{C}}\, p_B(v) &= p_{\mathcal{C}}(S\circ T(v)) \\ &= p_{\mathcal{C}}(S(T(v))) \\ &= [S]_D^{\mathcal{C}}\, p_D(T(v)) \\ &= [S]_D^{\mathcal{C}}[T]_B^D\, p_B(v)\end{aligned}$$

p_B is surjective, so every $x \in F^n$ has the form $p_B(v)$ and we may write

$$[S\circ T]_B^{\mathcal{C}}\, x = [S]_D^{\mathcal{C}}[T]_B^D\, x \quad \forall x \in F^n$$

Taking $x = e_j$, $1 \le j \le n$ we find that the j-th column of $[S\circ T]_B^{\mathcal{C}}$ equals the jth column of $[S]_D^{\mathcal{C}}[T]_B^D$

$\Rightarrow [S\circ T]_B^{\mathcal{C}} = [S]_D^{\mathcal{C}}[T]_B^D$.

<u>Ex.</u> Take $F = \mathbb{R}$, $V = W = P_2$

Let $T: P_2 \to P_2$ be the derivative.

Let $B = \{1, x, x^2\}$. By definition

$$[T]_B^B = [p_B(T(1)) \;\; p_B(T(x)) \;\; p_B(T(x^2))]$$
$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ - matrix form of the derivative!}$$

By thm MRLCT, $[T^2]_B^B = [T\circ T]_B^B = [T]_B^B [T]_B^B$.

Then the matrix of the second derivative is $([T]_B^B)^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

By def. $[T^2]_B^B = [p_B(T^2(1)) \;\; p_B(T^2(x)) \;\; p_B(T^2(x^2))]$
$$= \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Using this on $5 + x + 2x^2$:

$$p_B(5 + x + 2x^2) = \begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$$
$$p_B^{-1}\left(\begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}\right) = 1 + 4x.$$

<u>Thm</u> KNSI

Let $T \in LT(V, W)$, B a basis for V, $\mathcal{C}$ a basis for W.

Then $\ker T = N([T]_B^{\mathcal{C}})$ where the restriction of p_B to $\ker T$ is an isomorphism.

<u>Proof</u>:

Let $p_{B|}: \ker T \to F^n$ be the restriction of p_B to the subspace $\ker T \subseteq V$.

Prove that $p_B(v) \in N([T]_B^{\mathcal{C}})$ for every $v \in \ker T$.

By thm. FTMR

$$[T]_B^{\mathcal{C}}\, p_B(v) = p_{\mathcal{C}}(T(v)) = p_{\mathcal{C}}(0), \text{ since } v \in \ker T$$
$$= 0, \text{ since } p_{\mathcal{C}} \text{ is linear}$$

Thus $[T]_B^{\mathcal{C}}\, p_B(v) = 0$ i.e. $p_B(v) \in N([T]_B^{\mathcal{C}})$.

<u>Reminder</u>: $T \in LT(V, W)$ is injective $\Leftrightarrow \ker T = \{0\}$ where $\ker T = \{v \in V : T(v) = 0\}$

<u>Def.</u> Restriction to a subspace $U \subseteq V$ by $p_{B|}: U \to F^n$ where $p_{B|}(u) = p_B(u) \; \forall u \in U$ for $p_B: V \to F^n$.

$*$ Preserves linearity.

We already showed that p_{B1} is linear.

To show p_{B1} is injective, show $\ker(p_{B1}) = \{0\}$ (thm. KILT)

Suppose $p_{B1}(v) = 0$ for $v \in \ker T$

$\Rightarrow p_B(v) = 0$

$\Rightarrow v \in \ker(p_B) = \{0\}$ as p_B is injective

$\Rightarrow v = 0$.

Thus p_{B1} is injective.

Finally, to show p_{B1} is surjective, suppose $y \in N([T]_B^C) \subseteq F^n$.

$\Rightarrow [T]_B^C y = 0$

Since p_B surjective, $\exists v \in V$ such that $y = p_B(v)$

$\Rightarrow [T]_B^C p_B(v) = 0$

$\Rightarrow p_C(T(v)) = 0$ (thm. FTMR)

$\Rightarrow T(v) \in \ker p_C = \{0\}$ as p_C is injective

$\Rightarrow v \in \ker T$

$\Rightarrow p_{B1}(v) = y$ where $v \in \ker T$. $(p_{B1}(v) = p_B(v))$