Today: V has basis $\{v_1, \dots, v_n\}$
W has basis $\{w_1, \dots, w_m\}$
$T \in \mathcal{LT}(V, W)$

Thm. RCSI
T is surjective $\Leftrightarrow Col([T]_B^{\mathcal{C}}) = F^m$
Proof omitted.

Ex. Take $V = W = P_2$
Take $T =$ derivative. Let $B = \mathcal{C} = \{1, x, x^2\}$.
We already computed $[T]_B^B = \begin{bmatrix} 0&1&0\\0&0&2\\0&0&0 \end{bmatrix}$.
$Col([T]_B^B) = span\left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix} \right\} = \left\{ \begin{bmatrix}c_2\\2c_3\\0\end{bmatrix}, c_2, c_3 \in \mathbb{R} \right\} \neq \mathbb{R}^3$.
By theorem RCSI, T is not surjective

Recall: T is an isomorphism $\Leftrightarrow$ T is invertible $(T^{-1} \in \mathcal{LT}(W, V))$

Thm. IMR
T is an isomorphism $\Leftrightarrow [T]_B^{\mathcal{C}}$ is invertible.
Proof:
"$\Rightarrow$" Suppose T is an isomorphism.
Thus $V \cong W$ and $\dim V = \dim W$ (thm. IVSED)
We have $T \circ T^{-1} = I_W$ where $I_W(x) = x \quad \forall x \in W$
$\Rightarrow [T \circ T^{-1}]_{\mathcal{C}}^{\mathcal{C}} = [I_W]_{\mathcal{C}}^{\mathcal{C}}$
$\Rightarrow [T]_B^{\mathcal{C}} [T^{-1}]_{\mathcal{C}}^B = [I_W]_{\mathcal{C}}^{\mathcal{C}}$ (thm. MRCLT)
By definition, $[I_W]_{\mathcal{C}}^{\mathcal{C}} = [\rho_{\mathcal{C}}(I_W(w_1)) \cdots \rho_{\mathcal{C}}(I_W(w_m))]$
$= [\rho_{\mathcal{C}}(w_1) \cdots \rho_{\mathcal{C}}(w_m)]$ (since $I_W(w_j) = w_j$)
Since $w_1 = (1)w_1 + (0)w_2 + \dots + (0)w_m$, $\rho_{\mathcal{C}}(w_1) = \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = e_1$.
$\Rightarrow w_j = 1 w_j \Rightarrow \rho_{\mathcal{C}}(w_j) = e_j$.
$\Rightarrow [I_W]_{\mathcal{C}}^{\mathcal{C}} = I$
Thus $[T]_B^{\mathcal{C}} [T^{-1}]_{\mathcal{C}}^B = I$.
Similarily, starting with $T^{-1} \circ T$,
$\Rightarrow [T^{-1}]_{\mathcal{C}}^B [T]_B^{\mathcal{C}} = [I]_B^B$
$\Rightarrow [T^{-1}]_{\mathcal{C}}^B [T]_B^{\mathcal{C}} = I$
Thus $[T]_B^{\mathcal{C}}$ is invertible with $([T]_B^{\mathcal{C}})^{-1} = [T^{-1}]_{\mathcal{C}}^B$.
"$\Leftarrow$" Suppose $[T]_B^{\mathcal{C}}$ is invertible. Let $A = ([T]_B^{\mathcal{C}})^{-1}$ so that $A[T]_B^{\mathcal{C}} = I = [T]_B^{\mathcal{C}} A$.
All matrices here are $M_{nn}(F)$ so $\dim V = n = \dim W$.
Define $S: W \to V$ by $S(w) = \rho_B^{-1}(A \rho_{\mathcal{C}}(w)) \quad \forall w \in W$.
Recall that composing L.T. is linear, so S is linear.
Show S is T^{-1} by proving $\underline{S \circ T = I_V}$ and $T \circ S = I_W$ ← (assignment 7)
$S \circ T(v) = S(T(v)) = \rho_B^{-1}(A \underline{\rho_{\mathcal{C}}(T(v))}) = \rho_B^{-1}(A [T]_B^{\mathcal{C}} \rho_B(v))$
$= \rho_B^{-1}(I \rho_B(v))$
$= I_V(v)$.

$\Rightarrow S \circ T = I_V$.

Ex. As in previous example $V = W = P_2$, $B = \{1, x, x^2\}$

$[T]_B^B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$.

$\det([T]_B^B) = 0$

$\Rightarrow [T]_B^B$ is singular (non-invertible)

By thm IMR, T is not an isomorphism.

Section Change of Basis

Ex. Consider the case where $V = W = F^n$ and $B = \{v_1, \dots, v_n\}$, $\mathcal{C} = \{w_1, \dots, w_n\}$ are 2 bases for F^n.

We guess that $p_{\mathcal{C}}(I_{F^n}(v)) = [I]_B^{\mathcal{C}} p_B(v)$ $\forall v \in V$.

$$\begin{tikzcd} F^n \arrow[r, "I_{F^n}"] \arrow[d, "p_B"'] & F^n \arrow[d, "p_{\mathcal{C}}"] \\ F^n \arrow[r, "{[I]_B^{\mathcal{C}}}"'] & F^n \end{tikzcd}$$

$p_{\mathcal{C}}(I_{F^n}(v)) = [I_{F^n}]_B^{\mathcal{C}} p_B(v)$

$\Rightarrow p_{\mathcal{C}}(v) = [I_{F^n}]_B^{\mathcal{C}} p_B(v)$ where $[I_{F^n}]_B^{\mathcal{C}} = [p_{\mathcal{C}}(v_1) \cdots p_{\mathcal{C}}(v_n)]$

Def.

$[I]_B^{\mathcal{C}}$ is called the change of basis matrix. (from B to $\mathcal{C}$)

Ex. Take $V = W = \mathbb{R}^2$ and $B = \{e_1, e_2\}$, $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ $\qquad e_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Then $[I]_B^{\mathcal{C}} = [p_{\mathcal{C}}(e_1) \; p_{\mathcal{C}}(e_2)]$

$= \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$