Recall

$\mathbb{R}^n$ = n tuples of real numbers

$\mathbb{C}^n$ = n tuples of complex numbers

If $v = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n$, then $\bar{v} = \begin{bmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{bmatrix}$.

For $v, w \in \mathbb{C}^n$, then

$\overline{v+w} = \bar{v} + \bar{w}$

If $\alpha \in \mathbb{C}$, $v \in \mathbb{C}^n$

$\overline{\alpha v} = \bar{\alpha}\bar{v}$

Def. IP

The standard inner product on $\mathbb{C}^n$ is the map

$\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$

defined by

$$\langle u, v \rangle = \overline{[u]_1}[v]_1 + \ldots + \overline{[u]_n}[v]_n = \sum_{i=1}^n \overline{[u]_i}[v]_i.$$

The standard inner product on $\mathbb{R}^n$ is the map

$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

defined by

$$\langle u, v \rangle = \sum_{i=1}^n [u]_i [v]_i$$

Ex. $u = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}$, $\langle u, v \rangle = 1 \cdot 0 + (-1)(-1) + 4(-2) = -7$

Ex. $u = \begin{bmatrix} 1+i \\ 1-i \end{bmatrix}$, $v = \begin{bmatrix} i \\ 1 \end{bmatrix}$,
$$\langle u, v \rangle = \overline{(1+i)} \cdot i + \overline{(1-i)}(1) = (1-i)i + (1+i) = 2+2i$$

Note For any $u \in \mathbb{C}^n$,

$\langle u, 0 \rangle = \overline{[u]_1}0 + \ldots + \overline{[u]_n}0 = 0.$

Similarly, $\langle 0, u \rangle = 0 \quad \forall u \in \mathbb{C}^n$.

Remark

The standard inner product on $\mathbb{R}^n$ is sometimes called the dot product $\langle u, v \rangle = u \cdot v$.

We will not use it here.

The inner products agree on $\mathbb{R}^n$ so we do not give them different notations.

The textbook defines $\langle \cdot, \cdot \rangle$ on $\mathbb{C}^n$ with the complex conjugation in the first coordinate, but some (most) authors put the complex conjugation in the second coordinate.

Thm. IPVA

Suppose $u, v, w \in \mathbb{C}^n$. Then

① $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

② $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$.

Proof:

① By definition,

$$\begin{aligned}\langle u+v, w\rangle &= \sum_{j=1}^{n} \overline{[u+v]_j}\,[w]_j \\ &= \sum_{j=1}^{n} \left(\overline{[u]_j + [v]_j}\right)[w]_j \\ &= \sum_{j=1}^{n} \left(\overline{[u]_j} + \overline{[v]_j}\right)[w]_j \\ &= \sum_{j=1}^{n} \overline{[u]_j}[w]_j + \sum_{j=1}^{n} \overline{[v]_j}[w]_j \\ &= \langle u, w\rangle + \langle u, w\rangle.\end{aligned}$$

② Exercise. □

Thm. IPSM

Suppose $u, v \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$. Then

① $\langle \alpha u, v\rangle = \overline{\alpha}\langle u, v\rangle$

② $\langle u, \alpha v\rangle = \alpha\langle u, v\rangle$.

Proof:

① $$\begin{aligned}\langle \alpha u, v\rangle &= \sum_{j=1}^{n} \overline{[\alpha u]_j}\,[v]_j \\ &= \sum_{j=1}^{n} \overline{\alpha [u]_j}\,[v]_j \\ &= \sum_{j=1}^{n} \overline{\alpha}\,\overline{[u]_j}\,[v]_j \\ &= \overline{\alpha} \sum_{j=1}^{n} \overline{[u]_j}\,[v]_j \\ &= \overline{\alpha}\langle u, v\rangle.\end{aligned}$$

② Exercise. □

Remarks

① The version of thm. IPSM for $\mathbb{R}^n$ is

$\langle \alpha u, v\rangle = \alpha\langle u, v\rangle = \langle u, \alpha v\rangle$. $\forall \alpha \in \mathbb{R}, \forall u, v \in \mathbb{R}^n$.

The real inner product is linear in "each slot", ie. for fixed $u \in \mathbb{R}^n$, the map $v \mapsto \langle u, v\rangle$ is linear and for fixed v, $u \mapsto \langle u, v\rangle$ is linear

$\langle \cdot, \cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is an example of a billinear map; it is linear in each slot.

$\langle \cdot, \cdot\rangle : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}$ is linear in the second coordinate but only "half-linear" (sesquilinear) in the first coordinate.

Thm. IPAC

$\langle u, v\rangle = \overline{\langle v, u\rangle}$ $\forall u, v \in \mathbb{C}^n$

Proof:

$\langle u, v\rangle = \sum_{j=1}^{n} \overline{[u]_j}[v]_j = \overline{\left(\sum_{j=1}^{n} [u]_j \overline{[v]_j}\right)} = \overline{\langle v, u\rangle}$. □

Note. In $\mathbb{R}^n$, we can use the inner product to describe geometry (e.g. lengths of vectors, angles between vectors, etc.)

Length of $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \sqrt{a_1^2 + a_2^2}$

$\Rightarrow \left\| \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right\| = \sqrt{\left\langle \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \right\rangle}$

Def. NV

The norm (or length) of a vector $u \in \mathbb{C}^n$ or $u \in \mathbb{R}^n$ is defined by

$\|u\| = \sqrt{\langle u, u \rangle}$.

Note: In $\mathbb{C}^n$, $\langle u,u \rangle$ is always nonnegative.

Def.

An inner product on a complex vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ s.t.

1. $\langle u, v+w \rangle = \langle u,w \rangle + \langle u, v \rangle$
2. $\langle u, \alpha v \rangle = \alpha \langle u, v \rangle$
3. $\overline{\langle u, v \rangle} = \langle v, u \rangle$
4. $\langle u, u \rangle \in \mathbb{R}$ and nonnegative, and $\langle u, u \rangle = 0 \Leftrightarrow u = 0$.

Ex. $V = C[0,1] = \{ f : [0,1] \to \mathbb{R} : f \text{ is continuous} \}$

Define $\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx$.

Ex. $M_{nn}(\mathbb{C}) = \{ A : A \text{ is an } n \times n \text{ complex matrix} \}$.

Then $\langle A, B \rangle = \operatorname{tr}(A^* B)$ where $A^* =$ conjugate transpose of A.