MATH 2152 | Lecture 17 | 10 March 2026

Note: $F = \mathbb{R}$ or $\mathbb{C}$ for these inner products.

Def. Standard Inner Product

$\langle \cdot , \cdot \rangle : F^n \times F^n \to F$ is defined by

$$\langle u, v \rangle = \sum_{j=1}^{n} \overline{[u]_j}\,[v]_j \quad \forall u, v \in F^n.$$

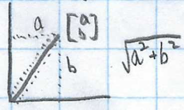
Def. Norm

$\| \cdot \| : F^n \to \mathbb{R}$ is defined by

$$\|u\| = \sqrt{\langle u, u \rangle} = \sqrt{\sum_{j=1}^{n} \overline{[u]_j}\,[u]_j} = \sqrt{\sum_{j=1}^{n} |[u]_j|^2} \qquad \forall u \in F^n.$$

Ex. In $\mathbb{R}^2$, a vector $u = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ has norm $\|u\| = \sqrt{\left\langle \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \right\rangle}$

$= \sqrt{a^2 + b^2} \geq 0.$



Thm. PIP

Suppose $u \in F^n$. Then $\langle u, u \rangle \geq 0$ and $\langle v, v \rangle = 0 \Leftrightarrow v = 0$.

Proof:

We compute $\langle u, u \rangle = \sum_{j=1}^{n} \overline{[u]_j}\,[u]_j = \sum_{j=1}^{n} |[u]_j|^2$.

Recall that $|[u]_j|^2 \geq 0$ for all $1 \leq j \leq n$. Since sum of nonnegative numbers is still negative, we obtain

$\sum_{j=1}^{n} |[u]_j|^2 \geq 0$.

For the second assertion, we compute

$\langle u, v \rangle = 0 \Leftrightarrow \sum_{j=1}^{n} |[u]_j|^2 = 0$

If $|[u]_1|^2 > 0$, then $\sum_{j=1}^{n} |[u]_j|^2 = |[u]_1|^2 + \sum_{j=2}^{n} |[u]_j|^2 > 0$.

This proves that if $\sum_{j=1}^{n} |[u]_j|^2 = 0$, then $|[u]_1|^2 = 0$

$\Rightarrow |[u]_1| = 0$

$\Rightarrow [u]_1 = 0$.

Similarily, the same argument holds for any $[u]_i$, $1 \leq i \leq n$.

Conversely, if $u = 0$ then $[u]_j = 0 \ \forall 1 \leq j \leq n$

$\Rightarrow \sum_{j=1}^{n} |[u]_j|^2 = 0$

$\Rightarrow \langle u, u \rangle = 0$.

Exercise

$\langle 0, v \rangle = 0$, and $\langle v, 0 \rangle = 0 \quad \forall v \in F^n$.

Ex. CNSN

Take $v = \begin{bmatrix} 3+2i \\ 1-6i \\ 2+4i \\ 2+i \end{bmatrix} \in \mathbb{C}^4$.

Then $\|v\|^2 = \langle v, v \rangle = 3^2 + 2^2 + 1^2 + 6^2 + 2^2 + 4^2 + 2^2 + 1^2 = 75$.

$\Rightarrow \|v\| = \sqrt{\langle v, v \rangle} = \sqrt{75} = 5\sqrt{3}$

Def.

The distance between $u, v \in \mathbb{C}^n$ is
$\|v-u\|$.

Ex. Take $u=\begin{bmatrix}1\\1\\1\\0\end{bmatrix}$, $v=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ in $\mathbb{R}^4$.
The distance between u and v is $\left\|\begin{bmatrix}1\\2\\3\\4\end{bmatrix}-\begin{bmatrix}1\\1\\1\\0\end{bmatrix}\right\| = \left\|\begin{bmatrix}0\\1\\2\\4\end{bmatrix}\right\| = \sqrt{0^2+1^2+2^2+4^2} = \sqrt{21}$

Exercise:

Distance between $u, v \in F^n$ is zero $\Leftrightarrow u=v$.

Thm.

Suppose $\alpha \in F$ and $v \in F^n$.
Then $\|\alpha v\| = |\alpha|\,\|v\|$.
Proof: exercise on assignment.

Remark:

Observe that for $u, v \in F^n$,
$\|v-u\| = \|(-1)(u-v)\| = |-1|\,\|u-v\| = \|u-v\|$.

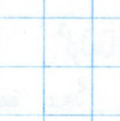
Thm.

Suppose $u, v \in F^n$. Then
$\|u+v\| \le \|u\| + \|v\|$.

Proof:

$\|u+v\|^2 = \langle u+v, u+v\rangle$
$= \langle u,u\rangle + \langle v,u\rangle + \langle u,v\rangle + \langle v,v\rangle$ (by unseen theorem)
$= \|u\|^2 + \overline{\langle u,v\rangle} + \langle u,v\rangle + \|v\|^2$
$= \|u\|^2 + 2\,\mathrm{Re}(\langle u,v\rangle) + \|v\|^2$
$\le \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2$
$\le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$ (by Cauchy-Scharz)
$= (\|u\| + \|v\|)^2$ (factoring)
$\Rightarrow \|u+v\| \le \|u\| + \|v\|$. ($\sqrt{x}$ is an increasing function)

Ex. Angles in $\mathbb{R}^2$



The cosine law: $c^2 = a^2+b^2-2ab\cos(\theta)$
In $\mathbb{R}^2$ the cosine law looks like
$\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\,\|v\|\cos(\theta)$
$\Rightarrow \langle u-v, u-v\rangle = \langle u,u\rangle + \langle v,v\rangle - 2\|u\|\,\|v\|\cos(\theta)$
$\Rightarrow 2\langle u,v\rangle = 2\|u\|\|v\|\cos(\theta)$
$\Rightarrow \cos(\theta) = \dfrac{\langle u,v\rangle}{\|u\|\|v\|}$ for $u \ne 0, v \ne 0$.
Thus the angle between 2 vectors is "built-in" to the inner product.

Def.

Two vectors $u, v \in F^n$ are orthogonal if $\langle u, v \rangle = 0$.

Exercise

$0 \in F^n$ is orthogonal to every vector in F^n.
Conversely, if $\langle v, u \rangle = 0 \ \forall u \in F^n$ then $v = 0$.
(Hint: take $u = v$ and use thm. PIP)

Ex. $\langle e_1, e_2 \rangle = (1)(0) + (0)(1) + 0 + \dots + 0 = 0$ are orthogonal.
More generally, $\langle e_i, e_j \rangle = 0$ for $1 \leq i \neq j \leq n$ in F^n.

Ex. $\left\langle \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -i \\ 2i \end{bmatrix} \right\rangle = (\overline{1})(1) + (\overline{i})(-i) + 0$

$= 1 + i^2$

$= 0.$

Thus the vectors are orthogonal.

Def.

A set $\{v_1, \dots, v_k\}$ if orthogonal if $\langle v_j, v_\ell \rangle = 0 \ \forall 1 \leq j \neq \ell \leq k$.

Ex. $\{e_1, \dots, e_n\} \subseteq F^n$ is an orthogonal set (orthogonal basis)
$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\}$ is an orthogonal set.