Determinant of a matrix

Recall F is a field (usually $\mathbb{R}$ or $\mathbb{C}$).

$M_{nn}(F)$ = vector space of n by n matrices with entries in F.

$A \in M_{nn}(F)$ is nonsingular if the only solution to the homogeneous linear system given by $\left[A \,\middle|\, \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix}\right]$ is $\begin{bmatrix}0\\ \vdots\\ 0\end{bmatrix} \in F^n$. (definition)

$A \in M_{nn}(F)$ is nonsingular

$\Leftrightarrow$ RREF of A is the identity matrix $I_n \in M_{nn}(F)$

$\Leftrightarrow$ Columns of A form a basis of F^n

$\vdots$

$\Leftrightarrow$ A is invertible. $(A^{-1}A = I_n = AA^{-1})$

Soon: $A \in M_{nn}(F)$ is nonsingular $\Leftrightarrow$ $\det(A) \neq 0$.

We need more background before we define $\det(A)$.

Recall: There are 3 row operations:

1. Swapping rows.
2. Multiply a row by $\alpha \in F$, $\alpha \neq 0$.
3. Multiply a row by $\alpha \in F$ and add it to another row.

We can recover each row operation by multiplying (on the left) by an elementary matrix.

Def. EM

1. Let $E_{i,j} \in M_{nn}(F)$ be the matrix obtained by swapping row i with row j in I_n $(1 \leq i \neq j \leq n)$.
   e.g. For $n=2$, $E_{1,2} = \begin{bmatrix}0&1\\1&0\end{bmatrix} = E_{2,1}$. For $n=3$, $E_{1,2} = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}$, $E_{1,3} = \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$.
2. $E_j(\alpha)$ = matrix obtained from multiplying row j of I_n by $\alpha \neq 0$.
   More explicitly, $[E_j(\alpha)]_{k\ell} = \begin{cases} 0 & \text{if } k \neq \ell \\ 1 & \text{if } k=\ell \text{ and } k \neq j \\ \alpha & \text{if } k=j=\ell. \end{cases}$
   For example, $E_2(4) \in M_{33}(\mathbb{R})$ is $E_2(4) = \begin{bmatrix}1&0&0\\0&4&0\\0&0&1\end{bmatrix}$.
3. For $\alpha \in F$ and $1 \leq i \neq j \leq n$, let $E_{i,j}(\alpha) \in M_{nn}(F)$ be the matrix obtained from multiplying row i of I_n by α and adding it to row j. For example, $E_{1,2}(4) \in M_{33}(\mathbb{R})$ equals $E_{1,2}(4) = \begin{bmatrix}1&0&0\\4&1&0\\0&0&1\end{bmatrix}$.

The matrices defined in 1-3 above are called elementary matrices.

Ex. Take $A = \begin{bmatrix}1&0&-1\\2&0&4\\-3&1&1\end{bmatrix}$.

We compute $E_{1,2}(4)A = \begin{bmatrix}1&0&0\\4&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&-1\\2&0&4\\-3&1&1\end{bmatrix} = \begin{bmatrix}1&0&-1\\6&0&0\\-3&1&1\end{bmatrix}$

Thm EMDRO

Suppose $A \in M_{nn}(F)$.

1. The matrix obtained by swapping rows i and j of A is equal to $E_{i,j}A$.
2. The matrix obtained by multiplying row j of A by $\alpha \neq 0$ is equal to $E_j(\alpha)A$.
3. The matrix obtained by multiplying row i of A by α and adding it to row j is $E_{i,j}(\alpha)A$.

Sketch of proof for 2:

Let $\alpha \in F$, $\alpha \neq 0$, $i \leq j \leq n$, $A \in M_{nn}(F)$.

$[E_j(\alpha)A]_{kk} = \sum_{t=1}^{n} [E_j(\alpha)]_{kt}[A]_{t\ell}$

$= 0 + \dots + 0 + [E_j(\alpha)]_{kk}[A]_{k\ell} + 0 + \dots + 0.$

If $k \neq j$ then $[E_j(\alpha)]_{kk} = 1$ and $[E_j(\alpha)A]_{kk} = 1[A]_{k\ell} = [A]_{k\ell}$.

If $k = j$ then $[E_j(\alpha)]_{kk} = \alpha$ and $[E_j(\alpha)A]_{j\ell} = \alpha[A]_{j\ell}$.

In the case $k \neq j$ the $k\ell$ entry of $E_j(\alpha)A$ is equal to the $k\ell$ entry of A. This means row k of $E_j(\alpha)A$ is equal to row k of A.

If $k = j$ then the $j\ell$ entry of $E_j(\alpha)A$ is α times the $j\ell$ entry of A. This means that the jth row of $E_j(\alpha)A$ is α times the jth row of A.

This shows that $E_j(\alpha)A$ is the matrix obtained by multiplying row j of A by α.

Thm. EMN

Elementary matrices are nonsingular.

Sketch of proof:

By thm. NME it suffices to prove that the elementary matrices are invertible.

Consider $E_{i,j} \in M_{nn}(F)$, $1 \leq i \neq j \leq n$.

By theorem ENDRO, $E_{i,j} = E_{j,i} = I_n$.

This shows that $(E_{i,j})^{-1} = E_{i,j}$ and so $E_{i,j}$ is invertible.

Next, consider $E_j(\alpha) \in M_{nn}(F)$, $\alpha \neq 0$.

By thm. EMDRO, $E_j(\alpha^{-1})E_j(\alpha) = I_n$

$\Rightarrow (E_j(\alpha))^{-1} = E_j(\alpha^{-1}) \Rightarrow E_j(\alpha)$ is invertible.

Finally, consider $E_{i,j}(\alpha) \in M_{nn}(F)$, $\alpha \in F$.

By thm. EMDRO, $E_{i,j}(-\alpha)E_{i,j}(\alpha) = I_n$

$\Rightarrow (E_{i,j}(\alpha))^{-1} = E_{i,j}(-\alpha)$

$\Rightarrow E_{i,j}(\alpha)$ is invertible.

Thm. NMPEM

$A \in M_{nn}(F)$ is nonsingular $\Leftrightarrow$ A is a product of elementary matrices.

Proof: "$\Leftarrow$" Suppose E_1 and E_2 are elementary matrices. Then by thm. EMN, E_1 and E_2 are nonsingular. By thm NPNT, E_1E_2 is nonsingular.

Exercise: Prove that if $E_1, \dots, E_k$ are elementary matrices then $E_1 \cdots E_k$ is nonsingular by induction on $k \geq 2$.