**Sidenote:** Inverses of Elementary Matrices

$E_{i,j} E_{i,j} = I_n$

$E_i(\alpha^{-1}) E_i(\alpha) = I_n$

$E_{i,j}(-\alpha) E_{i,j}(\alpha) = I_n$

**Thm.** NMPEH

Suppose $A \in M_{nn}(F)$. Then

$A$ is nonsingular $\Leftrightarrow$ $A$ is a product of elementary matrices.

Proof

We did "$\Leftarrow$" last lecture.

"$\Rightarrow$" Suppose $A \in M_{nn}(F)$ is nonsingular.

Then the RREF of $A$ is $I_n$. By thm. EMDRO, $E_k \cdots E_1 A = I_n$ for some elementary matrices $E_1, \ldots, E_k \in M_{nn}(F)$. Thm. EMN tells us that $E_1, \ldots, E_k$ are invertible.

Then we multiply both sides by the inverses of the elementary matrices,

$E_1^{-1} \cdots E_k^{-1} E_k \cdots E_1 A = E_1^{-1} \cdots E_k^{-1} I_n$

$\Rightarrow I_n \cdots I_n A = E_1^{-1} \cdots E_k^{-1} I_n$

$\Rightarrow A = E_1^{-1} \cdots E_k^{-1} I_n$

Since $E_1^{-1}, \ldots, E_k^{-1}$ are still elementary matrices (thm. EMN), we are done.

Definition of Determinant

**Def.** SM

$F$ is a field, $A \in M_{nn}(F)$.

Let $1 \le i, j \le n$. Then

$A(i|j) \in M_{n-1\,n-1}(F)$

is the matrix formed from $A$ by removing row $i$ and column $j$.

**Ex.** Take $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Then $A(1|1) = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$,

$A(2|1) = \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$

$A(2|3) = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$

**Def.** DM

Suppose $A \in M_{nn}(F)$.

If $n = 1$, then $\det(A) = [A]_{11}$

For $n \ge 2$, $\det(A) = \sum_{\ell=1}^{n} (-1)^{1+\ell} [A]_{1\ell} \det(A(1|\ell))$.

**Ex.** Take $A = [3] \in M_{11}(\mathbb{R})$. Then $\det(A) = 3$.

**Ex.** Take $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}(\mathbb{R})$. Then $\det(A) = [A]_{11} \det(A(1|1)) - [A]_{12} \det(A(1|2))$

$= ad - bc$

**Ex.** Take $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix} \in M_{33}(\mathbb{R})$

$\det(A) = (-1)^{1+1}(1) \det\left(\begin{bmatrix} 5 & 6 \\ 8 & 0 \end{bmatrix}\right) + (-1)^{1+2}(2) \det\left(\begin{bmatrix} 4 & 6 \\ 7 & 0 \end{bmatrix}\right) + (-1)^{1+3}(3) \det\left(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}\right)$

$= -48 + 82 - 9$

**Thm. DT**

Suppose $A \in M_{nn}(F)$.

Then $\det(A) = \det(A^T)$.

Transpose: $[A^T]_{ij} = [A]_{ji}$

Proof: by induction on $n \geq 1$.

Base case: $n = 1$.

$A = [A]_{11} = A^T$, therefore $\det(A) = \det(A^T)$.

Inductive assumption: assume holds for $n-1$.

Induction step: Prove for $A \in M_{nn}(F)$.

$$\begin{aligned}
\det(A^T) &= \frac{1}{n}\sum_{j=1}^{n} \det(A^T) \\
&= \frac{1}{n}\sum_{j=1}^{n}\sum_{\ell=1}^{n} (-1)^{j+\ell}[A^T]_{j\ell}\det(A^T(j|\ell)) \\
&= \frac{1}{n}\sum_{j=1}^{n}\sum_{\ell=1}^{n} (-1)^{j+\ell}[A]_{\ell j}\det(A(\ell|j)^T) \\
&= \frac{1}{n}\sum_{j=1}^{n}\sum_{\ell=1}^{n} (-1)^{j+\ell}[A]_{\ell j}\det(A(\ell|j)) \quad \text{(induction assumption)}
\end{aligned}$$

Hint: $A^T(i|j) = A(j|i)^T$ (no proof).

We need another theorem first!

**Thm. DER.**

Suppose $A \in M_{nn}(F)$ and $1 \leq j \leq n$. Then

$$\det(A) = \sum_{\ell=1}^{n}(-1)^{j+\ell}[A]_{j\ell}\det(A(j|\ell)).$$

Proof omitted.

Ex. Take $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \in M_{33}(\mathbb{R})$

Take $j = 2$. By thm. DER,

$$\det(A) = (-1)^{2+1}[A]_{21}\det(A(2|1)) + (-1)^{2+2}[A]_{22}\det(A(2|2)) + (-1)^{2+3}[A]_{23}\det(A(2|3))$$

$$= \dots$$

The determinant can be found from any row, not just the first as in the definition.

**Thm. DT**

Proof: continued.

$$\begin{aligned}
\det(A^T) &= \frac{1}{n}\sum_{j=1}^{n}\sum_{\ell=1}^{n}(-1)^{j+\ell}[A]_{\ell j}\det(A(\ell|j)) \\
&= \frac{1}{n}\sum_{\ell=1}^{n}\sum_{j=1}^{n}(-1)^{j+\ell}[A]_{\ell j}\det(A(\ell|j)) \\
&= \frac{1}{n}\sum_{\ell=1}^{n}\det(A) \quad \text{(thm. DER)} \\
&= \det(A). \quad \square
\end{aligned}$$

**Thm. DEC**

Suppose $A \in M_{nn}(F)$ and $1 \leq j \leq n$.

$$\det(A) = \sum_{\ell=1}^{n}(-1)^{j+\ell}[A]_{\ell j}\det(A(\ell|j))$$

Proof:

$$\begin{aligned}
\det(A) &= \det(A^T) && \text{(thm. DT)} \\
&= \sum_{\ell=1}^{n}(-1)^{\ell+j}[A^T]_{j\ell}A^T(j|\ell) && \text{(thm. DER)} \\
&= \sum_{\ell=1}^{n}(-1)^{\ell+j}[A]_{\ell j}\det((A(\ell|j))^T) \\
&= \sum_{\ell=1}^{n}(-1)^{j+\ell}[A]_{\ell j}\det(A(\ell|j)) && \text{(thm. DT)} \quad \square
\end{aligned}$$

<u>Ex.</u> Take $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ and $j = 3$.

Then $\det(A) = (-1)^{1+3}(3)\det\left(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}\right) + (-1)^{2+3}(6)\det\left(\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}\right) + (-1)^{3+3}(9)\det\left(\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}\right)$.

<u>Thm.</u> OZRC

If $A \in M_{n \times n}(F)$ has a zero row or column, then $\det(A) = 0 + \dots + 0 = 0$.

<u>Ex.</u> Take $A = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$.

$\det(A) = 0$, by thm. OZRC.

<u>Trivia</u>

$(-1)^{j+\ell} \det(A(\ell \mid j))$ is called the cofactor of row $\ell$ and col $j$ of $A$.