Recall:

For $A \in M_{nn}(F)$, $\det(A) \in F$

$\det(A) \neq 0 \iff A$ is nonsingular $\iff A$ is invertible $\iff A = E_1 \cdots E_k$

Elementary matrices: $E_{i,j}$, $E_j(\alpha)$, $E_{i,j}(\alpha)$

Thm. DRCS

Suppose $A \in M_{nn}(F)$ and set $B = E_{i,j}A$ for some $1 \leq i,j \leq n$. Then $\det(B) = -\det(A)$.

Sketch of proof:

First suppose $j = i+1$. Then $B(j|l) = B(i+1|l) = A(i|l)$ and $[B]_{i+1\,l} = [A]_{il}$

Expanding $\det(B)$ along row $j = i+1$, we obtain

$$\det(B) = \sum_{l=1}^{n} (-1)^{i+1+l} [B]_{i+1\,l} \det(B(i+1|l))$$
$$= (-1) \sum_{l=1}^{n} (-1)^{i+l} [A]_{il} \det(A(i|l))$$
$$= (-1) \det(A)$$

Exercise: Prove thm. DRCS by induction on $j - i \geq 1$ (where $j > i$)

Exercise: Prove that swapping columns of A gives a (-1) in the determinant.

Thm. DERC

Suppose $A \in M_{nn}(F)$ has 2 identical rows. Then $\det(A) = 0$.

Proof:

Suppose row i = row j in A, and $2 \neq 0$ in F. Then $E_{i,j}A = A$ and by thm. DRCS,

$$\det(A) = \det(E_{i,j}A) = -\det(A)$$
$$\Rightarrow 2\det(A) = 0$$
$$\Rightarrow \det(A) = 0.$$

Thm. Suppose $A \in M_{nn}(F)$ and $B = E_j(\alpha)A$ for some $\alpha \neq 0$, $1 \leq j \leq n$. Then $\det(B) = \alpha \det(A)$.

Proof: Expanding along row j,

$$\det(B) = \sum_{l=1}^{n} (-1)^{j+l} [B]_{jl} \cdot \det(B(j|l))$$
$$= \sum_{l=1}^{n} (-1)^{j+l} (\alpha [A]_{jl}) \cdot \det(A(j|l))$$
$$= \alpha \sum_{l=1}^{n} (-1)^{j+l} [A]_{jl} \cdot \det(A(j|l))$$
$$= \alpha \det(A)$$

Note that $B(j|l) = A(j|l)$ and $[B]_{jl} = \alpha [A]_{jl}$.

Thm. DRCMA

Let $A \in M_{nn}(F)$ and set $B = E_{i,j}(\alpha)A$, for some $1 \leq i \neq j \leq n$ and $\alpha \in F$. Then $\det(B) = \det(A)$.

Proof:

Let $C \in M_{nn}(F)$ be the matrix which replaces row j of A with row i of A. By construction, rows j and i of C are identical. So by thm. DERC, $\det(C) = 0$. Observe that $B(j|l) = A(j|l) = C(j|l)$.

$[B]_{jl} = [A]_{jl} + \alpha [A]_{il}$.

Then $\det(B) = \sum_{\ell=1}^{n} (-1)^{j+\ell} [B]_{j\ell} \det(B(j|\ell))$
$= \sum_{\ell=1}^{n} (-1)^{j+\ell} ([A]_{j\ell} + \alpha [A]_{i\ell}) \det(B(j|\ell))$
$= \sum_{\ell=1}^{n} (-1)^{j+\ell} [A]_{j\ell} \det(B(j|\ell)) + \alpha \sum_{\ell=1}^{n} (-1)^{j+\ell} [A]_{i\ell} \det(B(j|\ell))$
$= \sum_{\ell=1}^{n} (-1)^{j+\ell} [A]_{j\ell} \det(A(j|\ell)) + \alpha \sum_{\ell=1}^{n} (-1)^{j+\ell} [A]_{i\ell} \det(C(j|\ell))$
$= \det(A) + \alpha \sum_{\ell=1}^{n} (-1)^{j+\ell} [C]_{j\ell} \det(C(j|\ell))$
$= \det(A) + \alpha \det(C)$
$= \det(A) + \alpha \cdot 0$
$= \det(A)$. $\square$

<u>Thm.</u> DIM, DEM
$\det(I_n) = 1$
$\det(E_{i,j}) = -1$
$\det(E_j(\alpha)) = \alpha$
$\det(E_{i,j}(\alpha)) = 1$
<u>Sketch of proof:</u>
By assignment 1, $\det(I_n) = 1^n = 1$, as I_n is upper triangular.
Recall that $E_{i,j} = E_{i,j} I_n$. By theorem DRCS, $\det(E_{i,j}) = \det(E_{i,j} I_n) = -\det(I_n) = -1$.
Similarily, $E_j(\alpha) = E_j(\alpha) I_n$. By theorem DRCM, $\det(E_j(\alpha)) = \det(E_j(\alpha) I_n) = \alpha \det(I_n) = \alpha$.
Finally, $E_{i,j}(\alpha) = E_{i,j}(\alpha) I_n$. By theorem DRCMA, $\det(E_{i,j}(\alpha)) = \det(E_{i,j}(\alpha) I_n) = 1 \cdot \det(I_n) = 1$.

<u>Thm.</u> DEMMM
Suppose $A \in M_{nn}(F)$ and E is an elementary matrix. Then $\det(EA) = \det(E)\det(A)$.
<u>Proof:</u>
$\det(E_{i,j} A) = -\det(A)$ (thm DRCS) $= \det(E_{i,j}) \cdot \det(A)$ (thm DEM).
$\det(E_j(\alpha) A) = \alpha \det(A)$ (thm DRCM) $= \det(E_j(\alpha)) \cdot \det(A)$ (thm. DEM).
$\det(E_{i,j}(\alpha) A) = 1 \cdot \det(A)$ (thm DRCMA) $= \det(E_{i,j}(\alpha)) \cdot \det(A)$ (thm. DEM).

<u>Corollary</u>
Suppose $E_1, \dots, E_k \in M_{nn}(F)$ are elementary matrices and $A \in M_{nn}(F)$.
Then $\det(E_1 \cdots E_k A) = \det(E_1) \cdots \det(E_k) \det(A)$
<u>Proof:</u> by induction on k
Base case: $k=1$: Thm. DEMMM
Assume $\det(E_1 \cdots E_k A) = \det(E_1) \cdots \det(E_k) \det(A)$
Induction step: take $k+1$. $\det(E_1 \cdots E_{k+1} A) = \det(E_1 (E_2 \cdots E_{k+1} A))$
$= \det(E_1) \cdot \det(E_2 \cdots E_{k+1} A)$
$= \det(E_1) \cdot \det(E_2) \cdots \det(E_{k+1}) \cdot \det(A)$ (by inductive assumption)