Corollary:

Suppose $A \in M_{nn}(F)$ and $E_1, \dots, E_k \in M_{nn}(F)$ are elementary matrices.
Then $\det(E_1 \cdots E_k A) = \det(E_1) \cdots \det(E_k) \det(A)$.

Proof: by induction on $k \geq 1$

Base case: $k=1$, is theorem DEMMM.

Inductive assumption: assume true for k

Inductive step: prove for $k+1$

$$\det(E_1 \cdots E_{k+1} A) = \det(E_1) \det(E_2 \cdots E_{k+1} A)$$
$$= \det(E_1) \det(E_2) \cdots \det(E_{k+1}) \det(A).$$

Thm: SMZD

Suppose $A \in M_{nn}(F)$. Then

A is nonsingular $\Leftrightarrow \det(A) \neq 0$

Contrapositive: A is singular $\Leftrightarrow \det(A) = 0$.

Proof:

By thm. EMDRO,

$B = E_1 \cdots E_k A$

where B is the RREF of A, and $E_1, \dots, E_k$ are elementary matrices.

"$\Rightarrow$" Suppose A is nonsingular. Then $B = I_n$ (thm. NME).

So $1 = \det(I_n) = \det(B) = \det(E_1 \cdots E_k A)$ (*)

$= \det(E_1) \cdots \det(E_k) \det(A)$

By thm DEM, $\det(E_j) \neq 0 \ \forall 1 \leq j \leq k$.

Therefore $(\det(E_j))^{-1}$ exists $\forall 1 \leq j \leq k$.

Multiplying (*) by $\det(E_k)^{-1} \cdots \det(E_1)^{-1}$, we obtain

$\det(E_k)^{-1} \cdots \det(E_1)^{-1} = \det(A)$,

therefore $\det(A) \neq 0$.

"$\Leftarrow$" Conversely, suppose $\det(A) \neq 0$.

From thm. NME, if $B = I_n$ then A is nonsingular. So we will prove $B = I_n$.

We have $\det(B) = \det(E_1 \cdots E_k A)$

$= \det(E_1) \cdots \det(E_k) \det(A)$

Since $\det(E_j) \neq 0 \ \forall 1 \leq j \leq k$, and $\det(A) \neq 0$ the right hand side is $\neq 0 \Rightarrow \det(B) \neq 0$.

The definition of RREF implies that B is an upper triangular matrix. Therefore $0 \neq \det(B) = [B]_{11} \cdot [B]_{22} \cdots [B]_{nn}$.

Another consequence of RREF is that the diagonal entries $[B]_{jj}$ are either 1 or 0. Since $[B]_{11} \cdots [B]_{nn} \neq 0$, $[B]_{jj} = 1 \ \forall 1 \leq j \leq n$

$\Rightarrow B = I_n$ (since each 1 must be the only entry in its column to be in RREF).

Ex. Is $A = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & 1 \\ -1 & 2 & 5 \end{bmatrix} \in M_{33}(\mathbb{C})$ invertible?

$\det(A) = \sum_{\ell=1}^{3} (-1)^{\ell+1} [A]_{\ell 1} \det(A(\ell|1))$
$= (-1)^2(1)\det(\begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}) + 0 + (-1)^4(-1)\det(\begin{bmatrix} 3 & 0 \\ 4 & 1 \end{bmatrix})$
$= 20-2-3$
$= 15$
$\neq 0$

Therefore A is invertible (and nonsingular, etc.)

Note: Important facts about the determinant:

Let $A \in M_{nn}(F)$.

Let $A = [a_1 \cdots a_n]$ where $a_1, \cdots, a_n \in \mathbb{R}^n$ are the columns of A.

If $a_1 = \alpha b_1 + c_1$ for $b_1, c_1 \in \mathbb{R}^n$ and $\alpha \in F$.

Then $\det(A) = \det([a_1 \cdots a_n])$
$= \det([\alpha b_1 + c_1 \; a_2 \cdots a_n])$
$= \alpha \det([b_1 \; a_2 \cdots a_n]) + \det([c_1 \; a_2 \cdots a_n])$

$\Rightarrow \det$ is linear in the first column.

Let $A, B \in M_{nn}(F)$.

- $\det(A+B) \neq \det(A) + \det(B) \quad \forall A, B \in M_{nn}(F)$
- Cramer's rule:
- Define $[C]_{ij} = (-1)^{i+j} \det(A(i|j))$, $1 \leq i, j \leq n$.
  If A is invertible then $A^{-1} = \frac{1}{\det(A)} C$. (C is called the adjugate)

Thm. DRMM

Suppose $A, B \in M_{nn}(F)$. Then $\det(AB) = \det(A) \cdot \det(B)$.

Proof

Case 1: suppose A is nonsingular.

By thm. NMPEM, $A = E_1 \cdots E_k$ for some $E_1, \cdots, E_k \in M_{nn}(F)$ elementary matrices.

It follows that $\det(AB) = \det(E_1 \cdots E_k B)$
$= \det(E_1) \cdots \det(E_k) \det(B)$
$= \det(E_1) \cdots \det(E_k) \det(I_n) \det(B)$
$= \det(E_1 \cdots E_k I_n) \det(B)$
$= \det(E_1 \cdots E_k) \det(B)$
$= \det(A) \det(B)$.

Case 2: suppose A is singular.

Thm. NPNT says AB nonsingular $\Leftrightarrow$ A and B are nonsingular. The contrapositive: AB singular $\Leftrightarrow$ A or B is singular.

Since A is singular, AB is singular.

By thm. SMZD, $\det(AB) = 0$ and $\det(A) = 0$.

So $\det(AB) = 0 = \det(A)\det(B)$. $\square$

Review of Linear Transformations

Def.

Let V and W be vector spaces over a field F.

$T: V \to W$ is a linear transformation (or is linear) if

$T(u+v) = T(u) + T(v) \quad \forall u,v \in V$, and

$T(\alpha v) = \alpha T(v) \quad \forall \alpha \in F, v \in V$.

T is also linear $\Leftrightarrow T(\alpha u + v) = \alpha T(u) + T(v) \quad \forall \alpha \in F, u,v \in V$.

The set of all linear transformations from V to W is denoted by $\mathcal{LT}(V,W)$.

Thm.

Suppose $T \in \mathcal{LT}(V,W)$.

① $T(0) = 0$

② T injective $\Leftrightarrow \ker T = \{v : T(v) = 0\} = \{0\}$

③ T surjective $\Leftrightarrow T(V) = \{T(v) : v \in V\} = W$

④ If $V = F^n$ and $W = F^m$, every $T \in \mathcal{LT}(V,W)$ satisfies $T(v) = Av \ \forall v \in F$ for a unique matrix $A \in M_{mn}(F)$.
$A = [T(e_1) \cdots T(e_n)]$ where $\{e_1, \dots, e_n\}$ is the standard basis for F^n. (thm. MLCTV)

⑤ If $\{v_1, \dots, v_n\}$ is a basis for V, and $\{w_1, \dots, w_n\}$ is any set of vectors in W. Then there is a unique
$T \in \mathcal{LT}(V,W)$ such that $T(v_j) = w_j \ \forall 1 \le j \le n$. $(T(\alpha_1 v_1 + \dots + \alpha_n v_n) = \alpha_1 w_1 + \dots + \alpha_n w_n \ \forall \alpha_1, \dots, \alpha_n \in F)$

⑥ We can add linear transformations and multiply them by scalars so that $\mathcal{LT}(V,W)$ is a vector space over F.

Def.

A linear operator is a linear transformation from V to V. The vector space of linear operators is denoted by
$\mathcal{LT}(V,V) = \mathcal{LT}(V)$.