Eigenvalues and Eigenvectors

Ex. Suppose $A \in M_{nn}(F)$. Define $T \in \mathcal{LT}(V)$ by $T(v) = Av$ $\forall v \in F^n$.
What can T do to vectors v?
One possibility is that T may reduce to some kind of scaling.

Ex. If $A = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} \in M_{22}(\mathbb{R})$, then $T(e_1) = Ae_1 = \begin{bmatrix} -2 \\ 0 \end{bmatrix} = -2e_1$
$T(e_2) = Ae_2 = \begin{bmatrix} 0 \\ 4 \end{bmatrix} = 4e_2$,

which is scaling by different values in each row.

Ex. If $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \in M_{22}(\mathbb{R})$, then $T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = (-1)\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = (-2)\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

which also applies some sort of scaling to the vectors.
Not all linear operators behave this way, but we can find vectors upon which a linear operator acts by scalar multiplication. This is the point of eigenvectors & eigenvalues.

Def.
Suppose $A \in M_{nn}(F)$. Then a nonzero vector $v \in F^n$ satisfying $Av = \lambda v$ for some $\lambda \in F$ is called an eigenvector (of A) with eigenvalue $\lambda \in F$.

Thm. EMRCP
Suppose $A \in M_{nn}(F)$ and $v \neq 0$ is an eigenvector with eigenvalue λ, i.e. $Av = \lambda v$.
Then $\det(A - \lambda I_n) = 0$.
Conversely, if $\det(A - \lambda I_n) = 0$ for some $\lambda \in F$, then there exists $v \neq 0$ in F^n such that $Av = \lambda v$ and v is eigenvector with eigenvalue λ.

Proof:
$Av = \lambda v$ for $v \neq 0$
$\Leftrightarrow Av = \lambda I_n v$
$\Leftrightarrow Av - \lambda I_n v = 0$
$\Leftrightarrow (A - \lambda I_n)v = 0$
$\Leftrightarrow v \in \mathcal{N}(A - \lambda I_n)$
$\Leftrightarrow A - \lambda I_n$ is singular (def.)
$\Leftrightarrow \det(A - \lambda I_n) = 0$ (thm. SZRD)

Ex. PMS3
Take $A \in M_{33}(\mathbb{C})$ (from textbook). By thm. EMRCP, $\lambda \in \mathbb{C}$ is an eigenvalue of $A \Leftrightarrow \lambda$ is a solution to $\det(A - xI_3) = 0$.
$\det(A - xI_3) = \dots = 3 + 5x + x^2 - x^3$.
So the λ values are the roots to $3 + 5x + x^2 - x^3$.

Exercise:

$\det(A-xI_n)$ is a polynomial with coefficients in F of degree n, and has a leading coefficient (coefficient of x^n) of $(-1)^n$.

Def. CP

Suppose $A\in M_{nn}(F)$. Then the characteristic polynomial is $p_A(x)=\det(A-xI_n)$.

The characteristic equation is $p_A(x)=0$.

Thm. EMME

Every complex matrix $A\in M_{nn}(\mathbb{C})$ has a complex eigenvalue $\lambda\in\mathbb{C}$.

Proof:

$p_A(x)$ is a polynomial with complex coefficients. By thm EMRCP the eigenvalues of A are the roots of $p_A(x)$.

By the fundamental theorem of algebra, $p_A(x)$ has at least one root and at most n roots.

Ex. Consider $A=\begin{bmatrix}0&1\\-1&0\end{bmatrix}\in M_{22}(\mathbb{R})$.

Then $\det(A-xI_2)=\det\left(\begin{bmatrix}-x&1\\-1&-x\end{bmatrix}\right)=(-x)^2-(-1)(1)=x^2+1$.

Eigenvalues are real roots of $x^2+1=0$.

Since x^2+1 has no real roots, A has no eigenvalues.

(Trivia: A rotates $\mathbb{R}^2$ by $\frac{\pi}{2}$ counterclockwise).

Ex. PMS 3 continued

We had that $p_A(x)=3+5x+x^2-x^3$.

Factoring: $p_A(x)=-(x-3)(x-(-1))^2=0$ when $x=3$ or $x=-1$.

Thus A has eigenvalues 3 and -1. [after def. AME, we see that $\alpha_A(3)=1$ and $\alpha_A(-1)=2$.]

Def. AME

Suppose $\lambda\in F$ is an eigenvalue of $A\in M_{nn}(F)$. If $(x-\lambda)^k$ is the highest power of $(x-\lambda)$ in $p_A(x)$ then $k=\alpha_A(\lambda)$ is called the algebraic multiplicity of λ.

Thm. EMS

Suppose $\lambda\in F$ is an eigenvalue of $A\in M_{nn}(F)$, and let $\mathcal{E}_A(\lambda)=\{v\in F^n : Av=\lambda v\}$.

Then $\mathcal{E}_A(\lambda)$ is a subspace of F^n (called the λ-eigenspace of F^n)

Proof:

Using the subspace test, we see that $A0=0=\lambda 0$, so $0\in\mathcal{E}_A(\lambda)$.

If $v,w\in\mathcal{E}_A(\lambda)$ then

$A(v+w)=Av+Aw=\lambda v+\lambda w=\lambda(v+w)$, so $v+w\in\mathcal{E}_A(\lambda)$

Finally, for $\beta\in F$, $A(\beta v)=\beta(Av)=\beta(\lambda v)=\lambda(\beta v)$, and so $\beta v\in\mathcal{E}_A(\lambda)$.

By the subspace test, $\mathcal{E}_A(\lambda)$ is a subspace. $\square$

Def. EM, GME

Suppose $\lambda \in F$ is an eigenvalue of $A \in M_{nn}(F)$. Then $\dim \mathcal{E}_A(\lambda) = \gamma_A(\lambda)$ is called the geometric multiplicity of λ.