Recall:

$P_A(x) = \det(A - xI)$ is a polynomial

λ is an eigenvalue of A

$\Leftrightarrow$ λ is a root of $P_A(x)$

$\Leftrightarrow$ $P_A(\lambda) = 0$

$\Leftrightarrow$ $P_A(x) = (x-\lambda)^k q(x)$ for some $k \geq 1$ as big as possible.

The k above is called the algebraic multiplicity of $\lambda = \alpha_A(\lambda)$

If $F = \mathbb{C}$ then $p_A(x) = (-1)^n (x-\lambda_1)^{k_1}(x-\lambda_2)^{k_2} \cdots (x-\lambda_m)^{k_m}$

where $k_1 + \dots + k_m = n$.

λ-eigenspace $= \mathcal{E}_A(\lambda) = \{v \in F^n : Av = \lambda v\}$ is a subspace of F^n.

$= $ eigenvectors of A together with $0 \in F^n$.

The geometric multiplicity of λ is $\gamma_A(\lambda) = \dim(\mathcal{E}_A(\lambda))$.

Thm. EMNS

Suppose $A \in M_{nn}(F)$. Then

$\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I)$ (null space of $A - \lambda I \in M_{nn}(F)$)

Proof:

$v \in \mathcal{E}_A(\lambda)$ $\Leftrightarrow$ $Av = \lambda I_n v$

$\Leftrightarrow$ $(A - \lambda I_n)v = 0$

$\Leftrightarrow$ $v \in \mathcal{N}(A - \lambda I_n)$

Ex. ESMS3 (continued from CPMS3)

A has 2 eigenvalues since $p_A(x) = (-1)(x-3)(x-(-1))^2$.

$\alpha_A(3) = 1$, $\alpha_A(-1) = 2$. Let's compute $\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$ for $\lambda = 3, -1$.

$A - 3I_n$ has RREF $\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$ $[\lambda = 3]$

$\Rightarrow \mathcal{E}_A(3) = \left\{ \begin{bmatrix} -c_3/2 \\ c_3/2 \\ c_3 \end{bmatrix}, c_3 \in \mathbb{C} \right\}$

$= \operatorname{span}\left\{ \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow \gamma_A(3) = \dim(\mathcal{E}_A(3)) = 1$.

$A - (-1)I_n$ has RREF $\begin{bmatrix} 1 & 2/3 & 1/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $[\lambda = -1]$

$\Rightarrow \mathcal{E}_A(-1) = \left\{ \begin{bmatrix} (-2/3)c_2 + (-1/3)c_3 \\ c_2 \\ c_3 \end{bmatrix}, c_2, c_3 \in \mathbb{C} \right\}$

$= \operatorname{span}\left\{ \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\Rightarrow \gamma_A(-1) = \dim(\mathcal{E}_A(-1)) = 2$.

Ex. Take $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} \in M_{22}(\mathbb{C})$.

Then $P_A(x) = \det(A - xI_n) = \det\left(\begin{bmatrix} 3-x & 1 \\ 0 & 3-x \end{bmatrix}\right) = (3-x)(3-x) = (x-3)^2$

The only root of $p_A(x)$ is $\lambda = 3$. $\alpha_A(3) = 2$.

$\mathcal{E}_A(3) = \mathcal{N}(A - 3I_n) = \mathcal{N}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \left\{ \begin{bmatrix} c_1 \\ 0 \end{bmatrix} : c_1 \in \mathbb{C} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

$\gamma_A(3) = \dim \mathcal{E}_A(\lambda) = 1$.

Exercise

Take $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in M_{22}(\mathbb{C})$.

Then $p_A(x) = x^2 + 1 \Rightarrow A$ eigenvalues $\lambda = i, -i$.

$\Rightarrow p_A(x) = (x-i)^1 (x-(-i))^1$

$\Rightarrow \alpha_A(i) = 1 = \alpha_A(-i)$

$\mathcal{N}(A - iI_n) = \mathcal{N}\left(\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\right)$

RREF of $\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$ is $\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$.

$\Rightarrow \mathcal{E}_A(i) = \left\{ \begin{bmatrix} -ic_2 \\ c_2 \end{bmatrix} : c_2 \in \mathbb{C} \right\}$

$= \operatorname{span}\left\{ \begin{bmatrix} -i \\ 1 \end{bmatrix} \right\}$

$\Rightarrow \gamma_A(i) = 1 = \alpha_A(i)$

Similarly, $\gamma_A(-i) = 1 = \alpha_A(-i)$

Moreover, $\mathcal{E}_A(-i) = \operatorname{span}\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$.

Properties of eigenvalues and eigenvectors

Thm EDELI

Suppose $A \in M_{nn}(F)$ has distinct eigenvalues $\lambda_1, \dots, \lambda_k \in F$.

Let $v_1, \dots, v_k \in F^n$ be respective eigenvalues for $\lambda_1, \dots, \lambda_k$, e.g. $Av = \lambda_j \; \forall 1 \le j \le k$.

Then $\{v_1, \dots, v_k\}$ is linearly independent.

Proof: by induction on $k \ge 1$

Base case: $k = 1$

By definition, $v_1 \neq 0$ so $\{v_1\}$ is lin. independent.

Inductive hypothesis: assume theorem holds for $k-1$ distinct eigenvalues.

Inductive step: prove for k distinct eigenvalues.

Suppose $\beta_1 v_1 + \dots + \beta_k v_k = 0$ for some $\beta_1, \dots, \beta_k \in F$. (*)

Multiplying on the left by A, we obtain

$A(\beta_1 v_1 + \dots + \beta_k v_k) = A0$

$\Rightarrow \beta_1 A v_1 + \dots + \beta_k A v_k = 0$

$\Rightarrow \beta_1 \lambda_1 v_1 + \dots + \beta_k \lambda_k v_k = 0$

Multiply (*) by λ_k to obtain

$\lambda_k(\beta_1 v_1 + \dots + \beta_k v_k) = \lambda_k 0$

$\Rightarrow \beta_1 \lambda_k v_1 + \dots + \beta_k \lambda_k v_k = 0$ (2)

Subtracting equation (2) from equation (1) we obtain

$\beta_1(\lambda_1 - \lambda_k)v_1 + \dots + \beta_{k-1}(\lambda_{k-1} - \lambda_k)v_{k-1} + 0 = 0$.

Since $v_1, \dots, v_{k-1}$ are eigenvectors for distinct $\lambda_1, \dots, \lambda_{k-1}$, $\{v_1, \dots, v_{k-1}\}$ is lin. ind. (by inductive assumption).

By lin. ind., $\beta_1(\lambda_1 - \lambda_k) = \dots = \beta_{k-1}(\lambda_{k-1} - \lambda_k) = 0$

Since $\lambda_1 \neq \lambda_k$ (distinct) $\lambda_1 - \lambda_k \neq 0 \Rightarrow \beta_1 = 0$.

Similarly, $\beta_j = 0 \; \forall 1 \le j \le k-1$.

Returning to (*) we see that $0 + 0 + \dots + \beta_k v_k = 0$.

Since $v_k \neq 0$, $\beta_k = 0$. $\square$

Corollary

Suppose $A \in M_{nn}(F)$ has n distinct eigenvalues $\lambda_1, \dots, \lambda_n \in F$ and $v_1, \dots, v_K \in F^n$ are respective eigenvectors.
Then $\{v_1, \dots, v_n\}$ is a basis for F^n.

Proof:

By theorem EDELI, $\{v_1, \dots, v_n\}$ is lin. ind.
Since $n = \dim F^n$ theorem G tells us that $\{v_1, \dots, v_n\}$ is also a basis.

Thm SMZE

$A \in M_{nn}(F)$ is singular $\Leftrightarrow 0 \in F$ is an eigenvalue of A.

Contrapositive: A is nonsingular $\Leftrightarrow 0$ is not an eigenvalue of A.

Proof:

$$
\begin{aligned}
A \text{ singular } &\Leftrightarrow \det(A) = 0 \\
&\Leftrightarrow \det(A - 0 I_n) = 0 \\
&\Leftrightarrow 0 \text{ is a root of } p_A(x) \\
&\Leftrightarrow 0 \text{ is an eigenvalue of } A.
\end{aligned}
$$