Thm. SMZE

Suppose $A \in M_{nn}(F)$. Then

A is singular $\Leftrightarrow$ 0 is an eigenvalue of A

Contrapositive: A is nonsingular $\Leftrightarrow$ 0 is not an eigenvalue of A.

Proof:

A singular $\Leftrightarrow \mathcal{N}(A) \neq \{0\}$ (thm NME)

$\Leftrightarrow Av = 0$ for some $v \neq 0$

$\Leftrightarrow Av = 0v$ for some $v \neq 0$

$\Leftrightarrow 0$ is eigenvalue of A. □

Thm. ESMM

Suppose $\beta \in F$ and $A \in M_{nn}(F)$.

If v is an eigenvector of A with eigenvalue $\lambda \in F$, then v is an eigenvector of βA with eigenvector $\beta\lambda$.

Proof:

$$(\beta A)v = \beta(Av) = \beta(\lambda v) = (\beta\lambda)v.$$ □

Since v is an eigenvector of A, $v \neq 0$, thus v is an eigenvector of βA with corresponding eigenvalue $\beta\lambda$. □

Thm. EOMP

Suppose $A \in M_{nn}(F)$ has eigenvector $v \in F^n$ with eigenvalue $\lambda \in F$.

Then A^k has eigenvector v with eigenvalue λ^k for all $k \geq 1$.

Proof: by induction on $k \geq 1$.

Base case: $k=1$. $A^1 v = Av = \lambda v = \lambda^1 v$.

Inductive assumption: suppose $A^k v = \lambda^k v$.

Inductive step:

$$\begin{aligned} A^{k+1}v &= A(A^k v) \\ &= A(\lambda^k v) \quad \text{(inductive assumption)} \\ &= \lambda^k (Av) \\ &= \lambda^k(\lambda v) \\ &= \lambda^{k+1} v. \end{aligned}$$ □

Corollary EPM

Suppose $A \in M_{nn}(F)$ has eigenvector $v \in F^n$ with eigenvalue $\lambda \in F$.

Suppose further that $p(x) = \alpha_0 + \alpha_1 x + \dots + a_k x^k$ where $\alpha_1, \dots, \alpha_k \in F$.

Then $p(A)$ has eigenvector v with eigenvalue $p(\lambda)$.

Proof:

$$p(A)v = \left(\sum_{j=0}^{k} \alpha_j A^j\right)v = \sum_{j=0}^{k} \alpha_j A^j v = \sum_{j=0}^{k} \alpha_j \lambda^j v = \left(\sum_{j=0}^{k} \alpha_j \lambda^j\right)v = p(\lambda)v.$$ □

Thm. Cayley-Hamilton theorem

$p_A(A) = 0$ for any $A \in M_{nn}(F)$

Proof omitted, partially done in assignment 3.

Thm. ETM

Suppose $\lambda \in F$ is an eigenvalue of $A \in M_{nn}(F)$.

Then λ is an eigenvalue of A^T.

Proof:

Recall that $\det(A) = \det(A^T)$ (thm. DT)

$$\begin{aligned}\Rightarrow \det(A^T - xI_n) &= \det(A^T - (xI_n)^T) && (\text{since } xI_n \text{ is diagonal} \Rightarrow (xI_n)^T = xI_n)\\ &= \det((A + (-xI_n))^T) && (\text{since } A^T + B^T = (A+B)^T)\\ &= \det(A - xI_n) && (\text{since } \det(A) = \det(A^T))\end{aligned}$$

Recall that $\lambda \in F$ is an eigenvalue of $A^T \Leftrightarrow \det(A^T - \lambda I_n) = 0$.

By the above, λ is an eigenvalue of $A^T \Leftrightarrow \det(A^T - \lambda I_n) = 0$

$\Leftrightarrow \det(A - \lambda I_n) = 0$

$\Leftrightarrow \lambda$ is an eigenvalue of A. $\square$

Ex. Take $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix} \in M_{22}(\mathbb{R})$.

Then $\begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$A^T = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$

Then $\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \neq \lambda\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Therefore $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not an eigenvector for A^T.

However, $2 \in \mathbb{R}$ is still an eigenvalue of A^T.

To find eigenvectors for 2, compute $\mathcal{E}_{A^T}(2) = \mathcal{N}(A^T - 2I_n)$

Thm. EIM

Suppose $v \in F^n$ is an eigenvector of $A \in M_{nn}(F)$ with eigenvalue $\lambda \in F$.

If A is nonsingular, then v is an eigenvector of A^{-1} with eigenvalue λ^{-1}.

Proof:

Suppose A is nonsingular. Then A^{-1} exists, and $\lambda \neq 0$ (thm. SMZE).

$$\begin{aligned}Av = \lambda v &\Leftrightarrow A^{-1}Av = A^{-1}\lambda v\\ &\Leftrightarrow v = \lambda A^{-1}v\\ &\Leftrightarrow \lambda^{-1}v = \lambda^{-1}\lambda A^{-1}v && (\text{since } \lambda \neq 0,\ \lambda^{-1} \text{ exists})\\ &\Leftrightarrow \lambda^{-1}v = A^{-1}v\end{aligned}$$

Since v is an eigenvector for A, $v \neq 0$, thus v is an eigenvector for A^{-1} with eigenvalue λ^{-1}. $\square$

Thm ERMCP

Suppose $v \in \mathbb{C}^n$ is an eigenvector for $A \in M_{nn}(\mathbb{C})$ with eigenvalue $\lambda \in \mathbb{C}$.
Then $\bar{v}$ is an eigenvector of $\bar{A}$ with eigenvalue $\bar{\lambda}$.

Proof:

Taking the complex conjugate of $Av = \lambda v$, we obtain
$$\overline{Av} = \overline{\lambda v} \Leftrightarrow \bar{A}\bar{v} = \bar{\lambda}\bar{v}.$$
Since v is an eigenvector for A, $v \neq 0$, thus $\bar{v} \neq 0$, thus $\bar{v}$ is an eigenvector for $\bar{A}$ with eigenvalue $\bar{\lambda}$. $\square$

Corollary: ERMCP

Suppose $v \in \mathbb{C}^n$ is an eigenvector for a real matrix $A \in M_{nn}(\mathbb{C})$ with eigenvalue $\lambda \in \mathbb{C}$.
Then $\bar{v}$ is an eigenvector for $\bar{A} = A$ with eigenvalue $\bar{\lambda}$.

Ex. Recall $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in M_{22}(\mathbb{C})$ has eigenvector $\begin{bmatrix} i \\ 1 \end{bmatrix}$ with eigenvalue $-i$.
By ERMCP, $\begin{bmatrix} -i \\ 1 \end{bmatrix} = \overline{\begin{bmatrix} i \\ 1 \end{bmatrix}}$ is an eigenvector $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ with eigenvalue i.

Thm. DCP

The characteristic polynomial $p_A(x) = \det(A - xI_n)$ of $A \in M_{nn}(F)$ is a polynomial of degree n with leading coefficient $(-1)^n$.

Sketch of proof:

We prove by induction on ≥ 2 that $p_A(x) = ([A]_{11} - x)([A]_{22} - x) \cdots ([A]_{nn} - x) + p(x)$ $(*)$
where $p(x)$ is a polynomial of degree $\leq n-2$.

Base case: $n=2$. exercise.

Assume $(*)$ for $n-1$.

Inductive step: prove for n

- Expanding along the first row:

$$p_A(x) = \det(A - xI_n)$$
$$= \sum_{j=1}^{n} (-1)^{j+1} [A - xI_n]_{1j} \det((A - xI_n)(1|j))$$

First term:
$$\cancel{(-1)^2}[A - xI_n]_{11} \det((A - xI_n)(1|1)) = [A]_{11} - x \det((A - xI_n)(1|1))$$
$$= ([A]_{11} - x)([A]_{22} - x) + \dots + q(x) \text{ where } q \text{ has degree } n-3$$

Will finish proof next class.