Thm. DCP

The characteristic polynomial $p_A(x) = \det(A - xI)$ of $A \in M_{nn}(F)$ is a polynomial of degree n with leading coefficient $(-1)^n$.

Sketch of proof:

We prove by induction on $n \geq 2$ that

$(*)$ $p_A(x) = ([A]_{11} - x)([A]_{22} - x) \cdots ([A]_{nn} - x) + q(x)$ where $\deg(q(x)) \leq n-2$.

Base case: $n=2$, left as an exercise.

Assume $(*)$ is true for $n-1$.

Prove $(*)$ for n.

$$A - xI = \begin{bmatrix} [A]_{11} & \cdots & [A]_{1n} \\ \vdots & \ddots & \vdots \\ [A]_{n1} & \cdots & [A]_{nn} \end{bmatrix} + \begin{bmatrix} -x & 0 & \cdots & 0 \\ 0 & -x & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & -x \end{bmatrix} = \begin{bmatrix} [A]_{11} - x & \cdots & [A]_{1n} \\ \vdots & \ddots & \vdots \\ [A]_{n1} & \cdots & [A]_{nn} - x \end{bmatrix}$$

$$\begin{aligned} \det(A - xI) = & (-1)^{1+1}[A-xI]_{11} \det((A-xI)(1|1)) \\ & + (-1)^{1+2}[A-xI]_{12} \det((A-xI)(1|2)) + \cdots \\ & + (-1)^{1+n}[A-xI]_{1n} \det((A-xI)(1|n)) \end{aligned}$$

The first summand is $([A]_{11} - x) \det\left(\begin{bmatrix} [A]_{22} - x & \cdots \end{bmatrix}\right) = ([A]_{11} - x)\det(A(1|1) - xI)$, then by inductive assumption...

$= ([A]_{11} - x)\left[([A]_{22} - x) \cdots ([A]_{nn} - x) + q(x)\right]$ where $\deg(q(x)) \leq (n-1) - 2 = n-3$.

$= ([A]_{11} - x) \cdots ([A]_{nn} - x) + ([A]_{nn} - x) q(x)$ where $\deg(([A]_{nn} - x) q(x)) \leq n-2$.

The second summand is

$-[A]_{12} \det((A - xI)(1|2))$.

Degree of $\det((A-xI)(1|2))$ is less than or equal to $n-2$. (Assignment 4 proof)

Similarily, the degree of $(-1)^{1+l}[A]_{1l} \det((A-xI)(1|l)) \leq n-2$ for all $2 \leq l \leq n$.

Consequently the remaining terms with $l \geq 2$ do affect the degree or leading coefficient of the first term, and we see that:

$$\begin{aligned} \det(A - xI) &= ([A]_{11} - x) \cdots ([A]_{nn} - x) + q_1(x) + \text{polynomials of degree} \leq n-2 \\ &= ([A]_{11} - x) \cdots ([A]_{nn} - x) + q(x) \text{ with } \deg(q(x)) \leq n-2. \end{aligned}$$

Thm. NEM

Suppose $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of $A \in M_{nn}(\mathbb{C})$.

Then $\sum_{i=1}^{k} \alpha_A(\lambda_i) = n$.

Proof:

By the fundamental theorem of algebra,

$$\begin{aligned} p_A(x) &= \det(A - xI_n) \\ &= (-1)^n \prod_{i=1}^{k} (x - \lambda_i)^{\alpha_A(\lambda_i)} \end{aligned}$$

Comparing the degrees, we see that $n = \sum_{i=1}^{k} \alpha_A(\lambda_i)$. $\square$

Ex. Suppose $A \in M_{nn}(F)$ has distinct eigenvalues $\lambda_1, \dots, \lambda_k \in F$.
What does $p_A(x)$ look like?

$$p_A(x) = (-1)^n (x-\lambda_1)^{\alpha_A(\lambda_1)} \cdots (x-\lambda_k)^{\alpha_A(\lambda_k)} q(x)$$

where $\deg(q(x)) + \alpha_A(\lambda_1) + \dots + \alpha_A(\lambda_k) = n$, and $q(x)$ has no roots in F.

$\Rightarrow n \geq \alpha_A(\lambda_1) + \alpha_A(\lambda_2) + \dots + \alpha_A(\lambda_k) \geq k = \#$ eigenvalues.

Lemma

Suppose $A, B \in M_{nn}(F)$ and B is invertible.
Then (i) $\det(BAB^{-1}) = \det(A)$, and
(ii) $p_{BAB^{-1}}(x) = \det(BAB^{-1} - xI_n) = \det(A - xI_n) = p_A(x)$.

Proof:

(i) By thm DRMM,

$$\begin{aligned}\det(BAB^{-1}) &= \det(B)\cdot\det(A)\det(B^{-1}) \\ &= \det(B)\cdot\det(B^{-1})\cdot\det(A) \\ &= \det(BB^{-1})\cdot\det(A) \\ &= 1\cdot\det(A) \\ &= \det(A).\end{aligned}$$

(ii)
$$\begin{aligned}\det(BAB^{-1} - xI_n) &= \det(BAB^{-1} - xBB^{-1}) \\ &= \det(BAB^{-1} - B(xI_n)B^{-1}) \qquad (xBB^{-1} = BxB^{-1} = B(xI_n)B^{-1}) \\ &= \det(B(A - xI_n)B^{-1}) \\ &= \det(B)\det(A - xI)\det(B^{-1}) \\ &= \det(A - xI) \quad \square\end{aligned}$$

Thm ME

For every $A \in M_{nn}(F)$ and eigenvalue $\lambda \in F$ of A,

$$1 \leq \gamma_A(\lambda) \leq \alpha_A(\lambda) \leq n.$$

Sketch of proof:

If $\lambda \in F$ is an eigenvalue, then there is a nonzero eigenvector for λ and so $\mathcal{E}_A(\lambda) \neq \{0\}$
$\Rightarrow \gamma_A(\lambda) = \dim(\mathcal{E}_A(\lambda)) \geq 1$.

By comparing the degree of $p_A(x) = n$ with its $(x-\lambda)^{\alpha_A(\lambda)}$ factor, we see that $\alpha_A(\lambda) \leq n$.

To prove $\gamma_A(\lambda) \leq \alpha_A(\lambda)$, we have to find an invertible $B \in M_{nn}(F)$ such that

$$B^{-1}AB = \begin{bmatrix} \lambda & 0 & * \\ 0 & \lambda & \\ 0 & & * \end{bmatrix},$$ where λ appears $\gamma_A(\lambda)$ times.

Then $$p_A(\lambda) = p_{B^{-1}AB}(x) = \det(B^{-1}AB - xI_n) = \det\left(\begin{bmatrix} \lambda - x & 0 & \cdots & * \\ 0 & \lambda - x & & * \\ \vdots & & \ddots & \\ 0 & \cdots & & * \end{bmatrix}\right) = (\lambda - x)^{\gamma_A(\lambda)} q(x)$$

This means $(x-\lambda)$ appears at least $\gamma_A(\lambda)$ times in $p_A(x)$.

$\Rightarrow \alpha_A(\lambda) \geq \gamma_A(\lambda)$. (max # of times $(x-\lambda)$ appears in $p_A(x)$)