Thm ME

For every $A \in M_{nn}(F)$ and eigenvalue $\lambda \in F$ of A,
$1 \le \gamma_A(\lambda) \le \alpha_A(\lambda) \le n$

Sketch of proof:

Strategy: construct invertible B such that

$$B^{-1}AB = \left[\begin{array}{c|c} \begin{matrix}\lambda & & 0\\ & \ddots & \\ 0 & & \lambda\end{matrix} & * \\ \hline 0 & * \end{array}\right] \text{ where } \lambda \text{ appears } \gamma_A(\lambda) \text{ times.}$$

Then $p_A(x) = p_{B^{-1}AB}(x) = \det\left(\left[\begin{array}{c|c} \lambda \ddots \lambda & * \\ \hline 0 & * \end{array}\right]\right) = (x-\lambda)^{\gamma_A(\lambda)} q(x) \quad \Rightarrow \gamma_A(\lambda) \le \alpha_A(\lambda)$

Let $g = \gamma_A(\lambda)$. By def'n of dimension of $\mathcal{E}_A(\lambda)$ there exists a basis $\{u_1, \dots, u_g\}$ for $\mathcal{E}_A(\lambda)$. By thm. ELIS we can extend $\{v_1, \dots, v_g\}$ to a basis $\{v_1, \dots, v_g, w_1, \dots, w_k\}$ for F^n. $(g+k=n)$

Let $B = [v_1 \cdots v_g \ w_1 \cdots w_k] \in M_{nn}(F)$, i.e. the columns of B are the basis vectors.

Since the columns of B are linearly independent, B is invertible, by theorem NME.

Note that $I = B^{-1}B = B^{-1}[v_1 \cdots v_g \ w_1 \cdots w_k]$
$= [B^{-1}v_1 \cdots B^{-1}v_g \ B^{-1}w_1 \cdots B^{-1}w_k]$
$= [e_1 \cdots e_n]$ (since $I = [e_1 \cdots e_n]$)

$\Rightarrow e_1 = B^{-1}v_1,\ e_2 = B^{-1}v_2, \dots$

Note that Pe_j returns the j-th column of P.

We compute the j-th column of $B^{-1}AB$ for $1 \le j \le g$:

$B^{-1}AB\,e_j = B^{-1}Av_j$
$= B^{-1}\lambda v_j$ (since $v \in \mathcal{E}_A(\lambda)$)
$= \lambda B^{-1}v_j$
$= \lambda e_j$

Thus $B^{-1}AB = \left[\begin{array}{c|c} \lambda \ddots \lambda & * \\ \hline 0 & * \end{array}\right]$ as desired.

Similarity and Diagonalisation

Def

$A, B \in M_{nn}(F)$ are similar (A is similar to B) if there exists an invertible matrix $S \in M_{nn}(F)$ such that $A = SBS^{-1}$.

Ex. $\begin{bmatrix}5 & 0\\ 0 & -2\end{bmatrix}$ is similar to $\begin{bmatrix}1 & 3\\ 4 & 2\end{bmatrix}$ in $M_{22}(\mathbb{R})$, since
$\begin{bmatrix}20/7 & 20/7\\ 8/7 & -6/7\end{bmatrix}\begin{bmatrix}1 & 3\\ 4 & 2\end{bmatrix}\begin{bmatrix}3/4 & -1\\ 1 & 1\end{bmatrix} = \begin{bmatrix}5 & 0\\ 0 & -2\end{bmatrix}$ and $\begin{bmatrix}20/7 & 20/7\\ 8/7 & -6/7\end{bmatrix}^{-1} = \begin{bmatrix}3/4 & -1\\ 1 & 1\end{bmatrix}$.

Ex. If B is similar to $I_n \in M_{nn}(F)$, then $B = SI_nS^{-1} = SS^{-1} = I_n$.
The only matrix similar to I_n is I_n itself.

Thm SER

Suppose $A, B, C \in M_{nn}(F)$. Then

(i) A is similar to A (reflexive)

(ii) If A is similar to B, then B is similar to A (symmetric)

(iii) If A is similar to B, and B is similar to C, then A is similar to C (transitive)

Proof:

(i) $A = I_n A I_n^{-1}$

(ii) Suppose $A = SBS^{-1}$.

Then $S^{-1}AS = B$.

$\Rightarrow B = S^{-1}AS = S^{-1}A(S^{-1})^{-1}$

$\Rightarrow B$ is similar to A.

(iii) If $A = SBS^{-1}$, $B = PCP^{-1}$,

$\Rightarrow A = S(PCP^{-1})S^{-1} = (SP)C(P^{-1}S^{-1}) = (SP)C(SP)^{-1}$

$\Rightarrow A$ is similar to C.

Note: similarity is an equivalence relation

Thm. SMEE

If $A, B \in M_{nn}(F)$ are similar, then

$p_A(x) = p_B(x)$

and A and B have the same eigenvalues (thm. EMRCP)

Proved in earlier lemma.

Ex. $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ and $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ have the same eigenvalues (namely 1), however, $\begin{bmatrix}1&0\\0&1\end{bmatrix}$ is not similar to $\begin{bmatrix}1&1\\0&1\end{bmatrix}$.
(since $\begin{bmatrix}1&0\\0&1\end{bmatrix} = I_n$).

Def.

The matrix $A \in M_{nn}(F)$ is diagonalizable if A is similar to a diagonal matrix.
i.e. $A = SDS^{-1}$ where $[D]_{jk} = 0$ for $1 \le j \ne k \le n$ $\left(= S\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}S^{-1}\right)$

Ex. The earlier example showed that
$\begin{bmatrix}1&3\\4&2\end{bmatrix}$ is diagonalizable since $\begin{bmatrix}1&3\\4&2\end{bmatrix}$ is similar to $\begin{bmatrix}5&0\\0&-2\end{bmatrix}$.

Thm DC

A is diagonalizable $\Leftrightarrow$ there exists a basis $\{v_1, \dots, v_n\}$ of F^n consisting of eigenvectors of A
(i.e. $Av_j = \lambda_j v_j$ for some $\lambda_j \in F$).

Remark: Recall that every linear operator $T: F^n \to F^n$ is given by $T(v) = Av$ $\forall v \in F^n$ for some matrix $A \in M_{nn}(F)$.
If A is diagonalizable, then DC tells us that T is multiplying basis vectors by some scalars (eigenvalues).

Thm DC

Proof:

"$\Rightarrow$" Suppose $A = PDP^{-1}$ where $D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ is diagonal $\left([D]_{jk} = \begin{cases} 0 & j \neq k \\ \lambda_j & j = k. \end{cases} \right)$

This is equivalent to $AP = PD$ by multiplying by P on the right.

Let $P = [v_1 \cdots v_n]$, i.e. the j-th column of P is $v_j \in F^n$.

By thm. NME, $\{v_1, \dots, v_n\}$ is a basis for F^n.

Let's multiply AP and PD seperately.

$AP = A[v_1 \cdots v_n] = [Av_1 \cdots Av_n]$

$PD = P\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = P[\lambda_1 e_1 \cdots \lambda_n v_n]$

$= [P\lambda_1 e_1 \cdots P\lambda_n e_n]$

$= [\lambda_1 P e_1 \cdots \lambda_n P e_n]$

$= [\lambda_1 v_1 \cdots \lambda_n v_n]$.

Since $AP = PD$,

$[Av_1 \cdots Av_n] = [\lambda_1 v_1 \cdots \lambda_n v_n]$.

$\Rightarrow Av_1 = \lambda_1 v_1,\ Av_2 = \lambda_2 v_2, \dots$

$\Rightarrow v_j$ is an eigenvector with the eigenvalue λ_j.

"$\Leftarrow$" next lecture.